Consideration of Poisson's Ratio Effect in Structural Analysis Using Elements with Three Degrees of Freedom

by

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ABSTRACT

A simple and efficient technique for nonlinear analysis of structures was developed and verified by the authors. The results showed good agreement in following the initiation, location and propagation of cracks and in calculating failure load. One of the main factors which was not taken into account in the previous works using elements with three degrees of freedom, is Poisson's ratio. In this paper, we introduce a new technique to deal with this problem. This technique can be applied in both linear and nonlinear cases. We apply the new technique to compression test of specimens subjected to uniform compression stresses. We compare calculated Poisson's ratio from simulation result with theoretical one, and find good agreement between them. This indicates that the newly proposed method can be a good tool to simulate the behavior which is mainly governed by the Poisson's ratio like effect of confinement in concrete columns and/or soil.

INTRODUCTION

Although there are many phenomena in which the effects of Poisson's ratio are minor, in some cases, Poisson's ratio plays an essential role governing the phenomena. For example, effect of lateral confinement for columns subjected to axial forces is a typical case. This effect can be taken into account using Finite Element Method (FEM), however, fracture behavior can not be properly followed using the FEM especially after separation of structural members. Recently, many other methods were developed to deal with fracture behavior problems. These methods mainly assume that the elements are rigid, and hence, the Poisson's ratio effect is not taken into account. The Rigid Body and Spring Model, RBSM[1,2,3] and the Modified or Extended Distinct Element Method (MDEM or EDEM)[4,5] are the examples of these methods. Unfortunately, these methods have less accuracy comparing with the FEM, especially, in small deformation analysis. The MDEM or EDEM has better accuracy in large deformation analysis, where the structure members are totally separated and behave as "Discrete" elements in the space.

A new simple and accurate technique was developed by combining the advantages of both groups of the models, FEM and the other methods. It was proved in Ref. (6) that numerical simulation using the proposed method can be conducted till the structural failure in reasonable time with reliable accuracy. With the method, the study region is considered as an assembly of virtually divided small elements. These elements are connected by distributed springs in normal and tangential directions. The formulations for spring stiffness and spring failure conditions were developed so that the springs surrounding the element totally represent the element deformations and failure. The effects of Poisson's ratio were not taken into account in the formulation presented in Ref. (6). The objective of this paper is to introduce a technique to consider Poisson's ratio effects. The numerical results are compared with those obtained from theory and accuracy of the results is verified.

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ELEMENT FORMULATION

The two elements shown in Fig. (1) are assumed to be connected by pairs of normal and shear springs located at contact points which are distributed around the element edges. Each pair of springs totally represents stresses and deformations of a certain area of the studied elements. The total stiffness matrix is determined by summing up the stiffness matrices of individual spring around each element. Failure of springs is modeled by assuming zero stiffness for the spring being considered. Consequently, the developed stiffness matrix is an average stiffness matrix for the element according to the stress situation around the element. In the 2-dimensional model, three degrees of freedom are considered for each element. This leads to a relatively small stiffness matrix (size: 6x6). Stiffness matrix is developed for an arbitrary contact point with one pair of normal and shear springs as shown in Fig. (2). Equation (1) shows the upper-left quarter of stiffness matrix. In this formulation, the stiffness matrix of an element depends on the contact point location (distance L and the angles θ and α) and the stiffness of normal and shear springs which are determined according to the stress and strain situations at the contact point location. The spring stiffness is calculated simply by Eq. (2), where d is the distance between springs; T the thickness of the element; "a" the length of the representative area; E and G the Young's and shear modulus of the material, respectively.

$$\begin{bmatrix} \begin{matrix}\mathrm{Sin}^2(\theta+\alpha)K_n \\ +\mathrm{Cos}^2(\theta+\alpha)K_s\end{matrix} & \begin{matrix}-K_n\mathrm{Sin}(\theta+\alpha)\mathrm{Cos}(\theta+\alpha) \\ +K_s\mathrm{Sin}(\theta+\alpha)\mathrm{Cos}(\theta+\alpha)\end{matrix} & \begin{matrix}\mathrm{Cos}(\theta+\alpha)K_sL\mathrm{Sin}(\alpha) \\ -\mathrm{Sin}(\theta+\alpha)K_nL\mathrm{Cos}(\alpha)\end{matrix} \\ \begin{matrix}-K_n\mathrm{Sin}(\theta+\alpha)\mathrm{Cos}(\theta+\alpha) \\ +K_s\mathrm{Sin}(\theta+\alpha)\mathrm{Cos}(\theta+\alpha)\end{matrix} & \begin{matrix}\mathrm{Sin}^2(\theta+\alpha)K_s \\ +\mathrm{Cos}^2(\theta+\alpha)K_n\end{matrix} & \begin{matrix}\mathrm{Cos}(\theta+\alpha)K_nL\mathrm{Cos}(\alpha) \\ +\mathrm{Sin}(\theta+\alpha)K_sL\mathrm{Sin}(\alpha)\end{matrix} \\ \begin{matrix}\mathrm{Cos}(\theta+\alpha)K_sL\mathrm{Sin}(\alpha) \\ -\mathrm{Sin}(\theta+\alpha)K_nL\mathrm{Cos}(\alpha)\end{matrix} & \begin{matrix}\mathrm{Cos}(\theta+\alpha)K_nL\mathrm{Cos}(\alpha) \\ +\mathrm{Sin}(\theta+\alpha)K_sL\mathrm{Sin}(\alpha)\end{matrix} & \begin{matrix}L^2\mathrm{Cos}^2(\alpha)K_n \\ +L^2\mathrm{Sin}^2(\alpha)K_s\end{matrix} \end{bmatrix} \quad (1)$$

$$K_n = \frac{E*d*T}{a} \text{ and } K_s = \frac{G*d*T}{a} \quad (2)$$

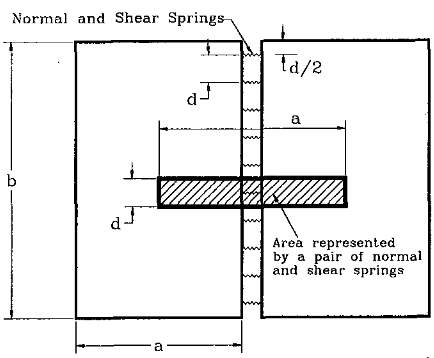


Fig. (1) Spring distribution and area of influence of each pair of springs

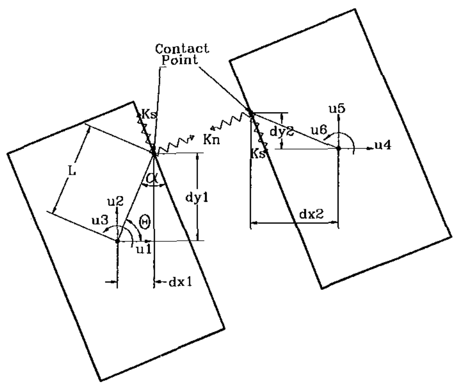


Fig. (2) Element shape, contact point and degrees of freedom

RIGID ELEMENTS AND FLEXIBLE ELEMENTS

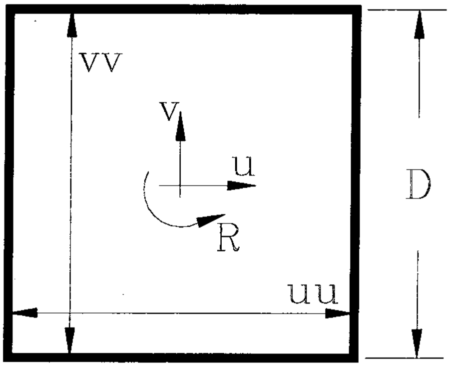


Fig. (3) Suggested degrees of freedom to consider Poisson's ratio effect

Referring to Fig. (1), vertical deformations do not affect the horizontal deformations. This indicates that elements are rigid and no lateral deformations are transformed through element edges. This is equivalent to having zero Poisson's ratio value. The main objective of this research is how to transform element behavior from rigid body state to a flexible state. Two main approaches can be adopted:

The first approach is by giving additional two degrees of freedom to the elements as shown in Fig. (3). The total number of degrees of freedom becomes 5 for each element. Three degrees of freedom (u, v and r) are for rigid body motion of the element and additional two components (uu and vv) are the relative deformations between the vertical and horizontal edges. The uu and vv correlate the deformations of element edges based on Poisson's ratio. Adding these effects to the stiffness matrix proposed in Eq. (1), the effects of Poisson's ratio can be considered. Although this approach is applicable, it has two main disadvantages. These disadvantages are:

- A coupling effect occurs between the degrees of freedom representing the rigid motion of elements and those representing the relative deformations between element edges. This effect makes the calculation of strains and stresses very difficult in comparison with the case where the effects of Poisson's ratio are not considered.
- The number of degrees of freedom increases from 3 to 5. This means that the time for assembling the stiffness matrix and for solving the equations gets to be more than double $[(5/3)^{\wedge 2}]$. Moreover, the requirement of computer memory capacity also drastically increases.

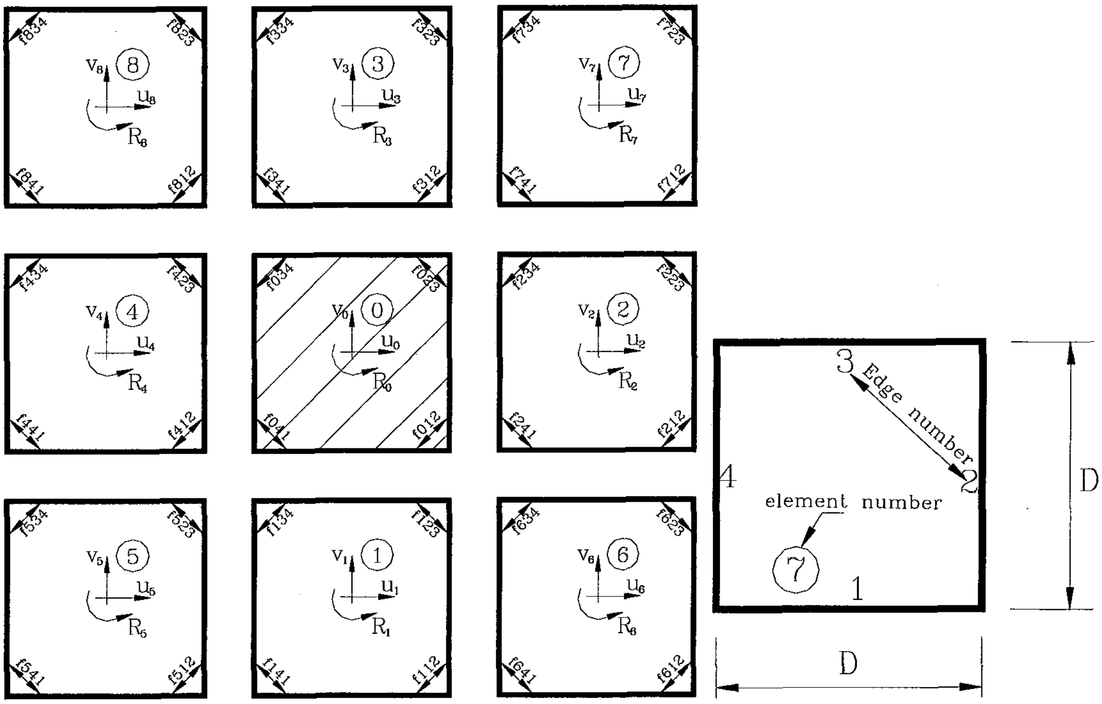
The second approach, adopted in this paper, is simple and does not have the disadvantages of the first approach. By the second approach, the effects of Poisson's ratio can be taken into account even when three degrees of freedom are used. Although the element motion is as rigid body, element shape is deformable. The idea is simply by correlating the stiffness matrix of each element edges by those of adjacent edges. The idea is illustrated in the next sections.

GEOMETRICAL DEFINITIONS

Assume the elements arrangement as shown in Fig. (4). As a general case, the element number (0) is surrounded by eight elements. Each element has three degrees of freedom and four edges numbered from 1 to 4, refer to Fig. (4-b). The factors "f" as shown in Fig. (4-a) represent the element continuity which will be illustrated in the following sections:

$$fijk=fij*fik \qquad (3)$$

Where, "i" is the element number, j and k represent edge numbers. The factor "fij" is equal to one if connecting springs of element "i" at the edge "j" exist, like "f52" and "f53". If the element at edge "j" is an edge element or the springs are failed because of cracks, "fij" is set to zero, like "f51" and "f54". This means that "fijk" is equal to one when the connection of the element "i" at the edges "j" and "k" is valid, like "f523". Otherwise, "fijk" is equal to zero, "f512", "f534" and "f541". These factors are used to develop the stiffness matrix which enables us to take into account the effects of Poisson's ratio in general for any element configuration.



(a) (b)

Fig. (4) Element configuration and continuity factors definition

NUMERICAL ANALYSIS PROCEDURE

The original stiffness matrix, which does not take into account of the effects of Poisson's ratio, is developed by summing the stiffness matrices of springs around each element. Additional terms are given to consider the Poisson's ratio. For each degree of freedom, the additional stiffness matrix terms are developed by assuming a corresponding displacement in the direction of the studied degree of freedom and calculating the straining action at the centroid of each surrounding element. Figure (5) shows the additional terms corresponding to horizontal displacement of the element (0). Figure (6) deals with vertical displacement and Figs. (7-a) and (7-b) deal with rotational degree of freedom. These terms are added directly to the global stiffness matrix.

Referring to Fig. (5), horizontal displacement is applied to the element (0). All other degrees of freedom are restrained. Springs between the elements (0 and 2) are subjected to compression. The area represented by the compressed springs, group "A", is the right half of the element (0) and the left half of the element (2). Compression forces (horizontal) between the elements (0 and 2) leads to lateral (vertical) displacement for the area represented, group "A\"". This lateral displacement is a function of Poisson's ratio. As this lateral displacement is prevented, because all degrees of freedom of other elements are assumed to be restrained, additional stresses exist in zone "A\"". These additional stresses are assumed to be uniform over the element edges. The additional stresses are transformed to the center of elements as force and moment, and added to the global stiffness matrix. When one of surrounding elements does not exist, the calculated secondary stresses change. For example, stress components between the elements (2 and 6) exist if the connection between the elements (0 and 2) and (2 and 6) are valid. This means that elements exist and springs are not cracked. For example, the element (6) does not exist, no additional stress component on the element (2) is used. This means that "f21" (element 2 and edge 1) is equal to zero. The stiffness matrix components corresponding to the applied unit displacements are determined by transmitting the calculated secondary stresses to the centroids of the corresponding elements.

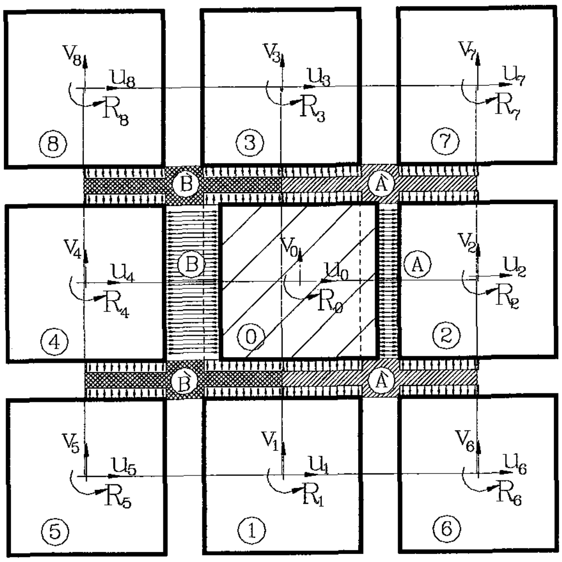


Fig. (5) Secondary stresses due to horizontal displacement of the element (0)

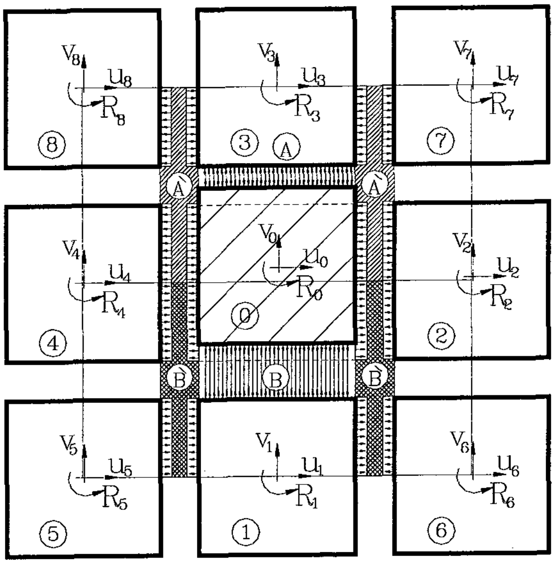
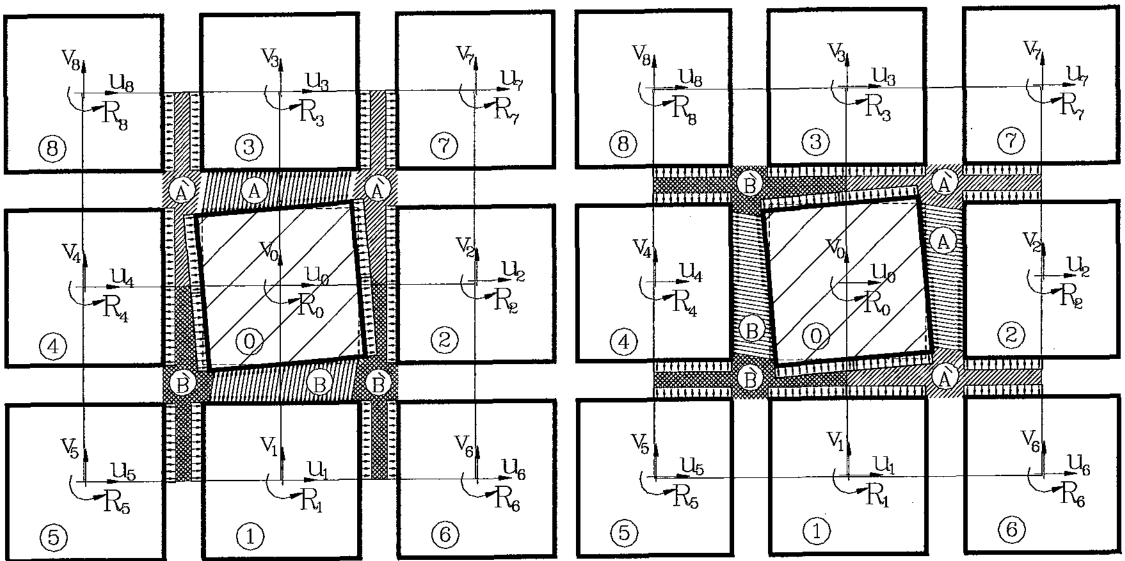


Fig. (6) Secondary stresses due to vertical displacement of the element (0)



(a) Effect of rotation of horizontal edges on stresses of vertical edges

(b) Effect of rotation of vertical edges on stresses of horizontal edges

Fig. (7) Secondary stresses due to rotational displacement of the element (0)

Table (1) Added stiffness matrix values due to unit displacement of each degree of freedom of element (0)

(0)			(1)			(2)		
u0	**v0**	**r0**	**u1**	**v1**	**r1**	**u2**	**v2**	**r2**
0	p0* (-f012 +f023 -f034 +f041)	m0* (-f012 +f023 +f034 -f041)	0	p0* (+f012 -f041)	m0* (+f012 +f041)	0	p2* (-f241 +f234)	m2* (+f241 -f234)
p0* (-f012 +f023 -f034 +f041)	0	m0* (-f012 -f023 +f034 +f041)	p1* (-f123 +f134)	0	m1* (+f123 -f134)	p0* (+f012 -f023)	0	m0* (+f012 +f023)
m0* (-f012 +f023 +f034 -f041)	m0* (-f012 -f023 +f034 +f041)	-2* m0*D/4 (f012 f023 f034 f041)	m1* (-f123 -f134)	m0* (+f012 -f041)	D/4* (+m0* f012 +m0* f041 +m1* f123 +m1* f134)	m0* (f012 -f023)	m2* (-f241 -f234)	D/4* (+m2* f241 +m2* f234 +m0* f012 +m0* f023)

(3)			(4)			(5)		
u3	**v3**	**r3**	**u4**	**v4**	**r4**	**u5**	**v5**	**r5**
0	p0* (-f023 +f034)	m0* (-f023 -f034)	0	p4* (+f412 -f423)	m4* (+f412 -f423)	0	-p4* f412	-m4* f412
p3* (+f312 -f341)	0	m3* (+f312 -f341)	p0* (-f041 +f034)	0	m0* (-f041 -f034)	-p1* f134	0	m1* f134
m3*(+f312 +f341)	m0*(+f023 -f034)	D/4* (+m0* f023 +m0* f034 +m3* f312 +m3* f341)	m0* (+f041 -f034)	m4* (+f412 +f423)	D/4* (+m4* f412 +m4* f423 +m0* f041 +m0* f034)	m1* f134	m4* -f412	D/4* (-m4* f412 -m1* f134)

(6)			(7)			(8)		
u6	**v6**	**r6**	**u7**	**v7**	**r7**	**u8**	**v8**	**r8**
0	+p2* f241	-m2* f241	0	-p2* f234	+m2* f234	0	+p4* f423	+m4* f423
+p1* f112	0	-m1* f112	-p3 f312	0	-m3* f312	+p3 f341	0	+m3* f341
+m1* f112	+m2* f241	D/4* (-m2* f241 -m1* f112)	-m3* f312	+m2* f234	D/4* (-m2* f234 -m3* f312)	-m3* f341	-m4* f423	D/4* (-m4* f423 -m3* f341)

Springs between the elements (0 and 4) are subjected to tension, group "B". The area represented by the springs subjected to tension is the left half of the element (0) and the right half of the element (4). Tension forces (horizontal) between the elements (0 and 4) leads to lateral (vertical) displacement for the area represented, group "B\"". Additional stresses exist in zone "B\"".

The calculated additional stresses are based only on assumption that the element (0) is surrounded from all directions by elements. For example, if the element (8) does not exist, no stress component exists between the elements (4 and 8). This means that lateral displacement is permitted in the upper edge of the element (4). To make the technique general, the calculated reaction component is multiplied by a continuity factor representing the continuity condition of each element edge.

Referring to Fig. (6), the same technique can be applied to the vertical displacement component of the element (0). Figure (7-a) shows the effect of rotation of horizontal edges of the element (0) while Fig. (7-b) shows the effect of rotation of vertical edges. In all the cases, secondary stresses, calculated in case of element rotation, are assumed to be uniformly distributed over the element edges.

Global stiffness matrix should be modified to add the consideration of the effects of the Poisson's ratio. These effects are shown in Table (1). The effects of Poisson's ratio due to unit displacement of each degree of freedom of the element (0) on the other surrounding elements are shown in the table.
The transmitted force and moments "p_i" and "m_i" are defined as:

$$p_i = \frac{\nu * E_i * t_i}{4} \quad \text{and} \quad m_i = \frac{\nu * E_i * t_i}{4} * \frac{D}{4} \qquad (4)$$

Where, ν is Poisson's ratio; E the Young's modulus; t the element thickness; D the element size. The subscript "i" is the element number.

It can be noticed from Table (1) that the stiffness matrix is general for any element configuration due to the use of element continuity factors. If one of neighboring elements does not exist, its effect on the stiffness matrix is automatically removed.

Calculation algorithm of strains is different from that in case of perfectly rigid elements. In case of perfectly rigid elements, strains are calculated directly from the relative displacement of spring ends. When the effects of Poisson's ratio effects are considered, the following technique is adopted after assembling the global stiffness matrix and solution of equations:

1. For each spring, calculate the apparent strain from the displacements of spring ends, ε'_x and ε'_y by:

$$\varepsilon'_x = \frac{d_x}{D} \quad \text{and} \quad \varepsilon'_y = \frac{d_y}{D} \qquad (5)$$

   Where, d_x and d_y are relative displacements of spring ends in x and y directions, respectively. For each spring, either ε'_x or ε'_y can be calculated. The same formulae are used for calculation of strain in case of perfectly rigid elements.

2. Calculate the average strain for each element in x and y directions, ε'_{xa} and ε'_{ya}.
3. Calculate the true strain for each spring in x and y directions, ε_x and ε_y, from:

$$\varepsilon_x = \varepsilon'_x + \upsilon\varepsilon'_{ya} \quad \text{and} \quad \varepsilon_y = \varepsilon'_y + \upsilon\varepsilon'_{xa} \qquad (6)$$

VERIFICATION OF THE PROPOSED TECHNIQUE

The numerical technique proposed is verified by analyzing elastic prism specimen subjected to uniform compression. Simulations are carried out using two models with and without friction between loading plates and specimen. The size of the specimens is (20x20x40) cm. The applied stresses in all cases are the same. The Poisson's ratio is varied from 0 to 0.5 with 0.1 increment. The friction coefficient adopted is 0.4.

Figure (8) shows the deformed shape in cases where no friction exists between the loading plates and the specimen. Displacements are generally uniform in the specimen. Increasing the Poisson's ratio leads to increasing both the vertical and horizontal displacements. From the simulated vertical and horizontal displacements, Poisson's ratios are calculated and compared with the theoretical ones in Fig. (9). The results show good agreement with theoretical results and the maximum difference between the theoretical and calculated values is less than 1%. This gives evidence that the numerical technique proposed is accurate enough and can be applied to structural phenomena governed by the Poisson's ratio.

Figure (10) shows the deformed shapes in cases of the specimen having different Poisson's ratios where friction exists. In case when the Poisson's ratio, ν, equals to zero, no horizontal displacement exists. On the other hand, lateral displacement is maximum when ν is equal to 0.5. Moreover, the vertical displacement becomes maximum when ν is 0.5 and minimum when ν is 0. The effect of friction between the loading plates and the specimen is also obvious.

Figure (11) shows the stress contours for (σ_x, σ_y, τ_{xy}) in case where ν is equal to 0.5. For σ_y, it is obvious that the effect of confining stresses leads to increasing the vertical stress near the center of the specimen and reducing the vertical stress near the prism edges of the specimen. However, the effect of Poisson's ratio on vertical stresses is not large. In case of σ_x, it is obvious that the effect of confining stress is maximum near the loading plates, because of friction, and this effect vanishes in the middle of the specimen. Shear stresses, τ_{xy}, is maximum near the loading plates, because of friction, and this effect disappears at the center line of the beam and away from the loading plates.

CONCLUSION

We introduced a simple and efficient technique to deal with the effects of the Poisson's ratio. The idea of this technique mainly depends on how to correlate element deformations with the three degrees of freedom used for each element motion. We verified the numerical results obtained by comparing with the theoretical ones and we got good agreement. Advantages of the proposed technique are listed below:

1. The technique is general and it can be applied to any structural phenomenon or geometry.
2. Three degrees of freedom were assumed for each element and no additional degrees of freedom were adopted to deal with this new factor.
3. The CPU time for solving equations is not highly affected because of no additional degrees of freedom. Only the time of assembling the global stiffness matrix increases. This increase is a function of the number of elements, not the number of springs. Moreover, no additional computer memory capacity is required as the dimension of matrices does not change.
4. The technique can be easily applied to the effects of reinforcement bars and different material.
5. With the technique, simulation can be performed in both linear and nonlinear cases, and also before and after generation of cracks.

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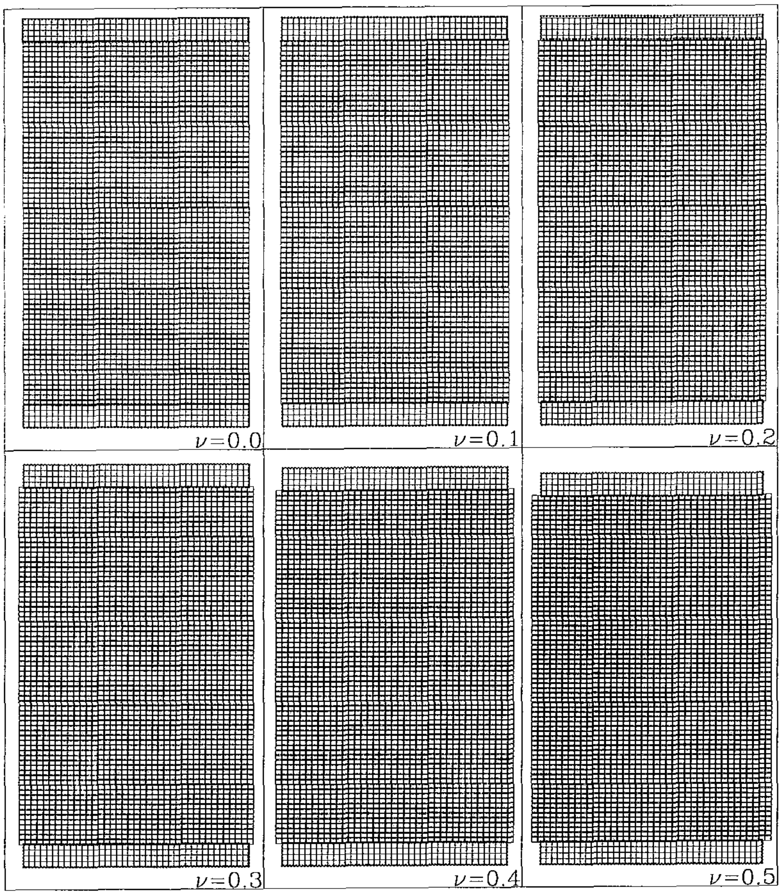

Fig. (8) Deformed shapes of specimens with different Poisson's ratio under uniform compression (without friction, Scale Factor=50)

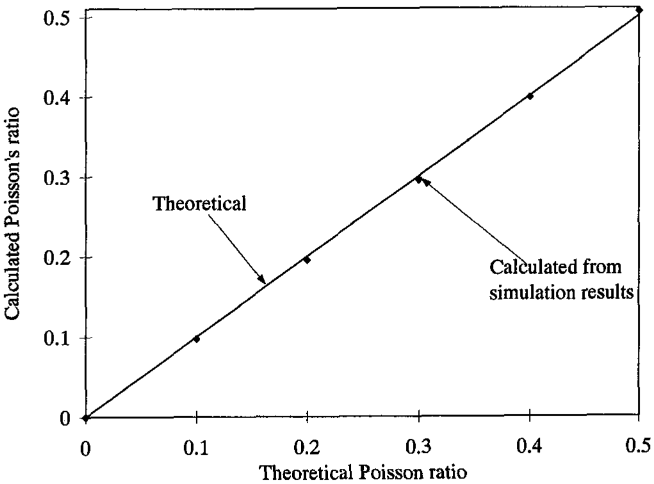

Fig. (9) Comparison between theoretical and calculated Poisson's ratios

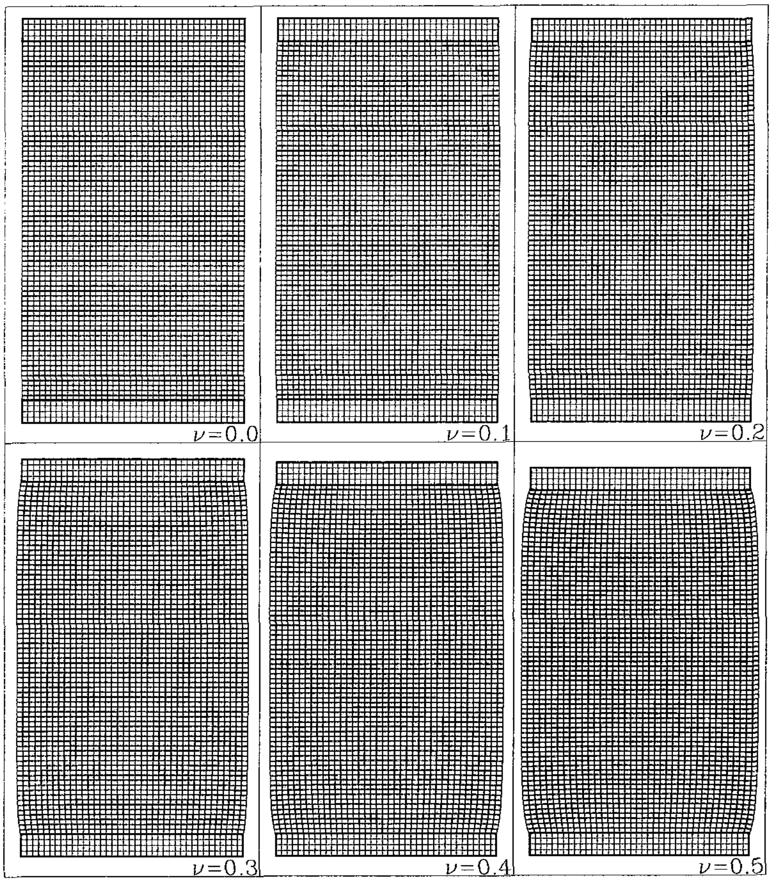

Fig. (10) Deformed shapes of specimens with different Poisson's ratio under uniform compression (with friction, Scale Factor=50)

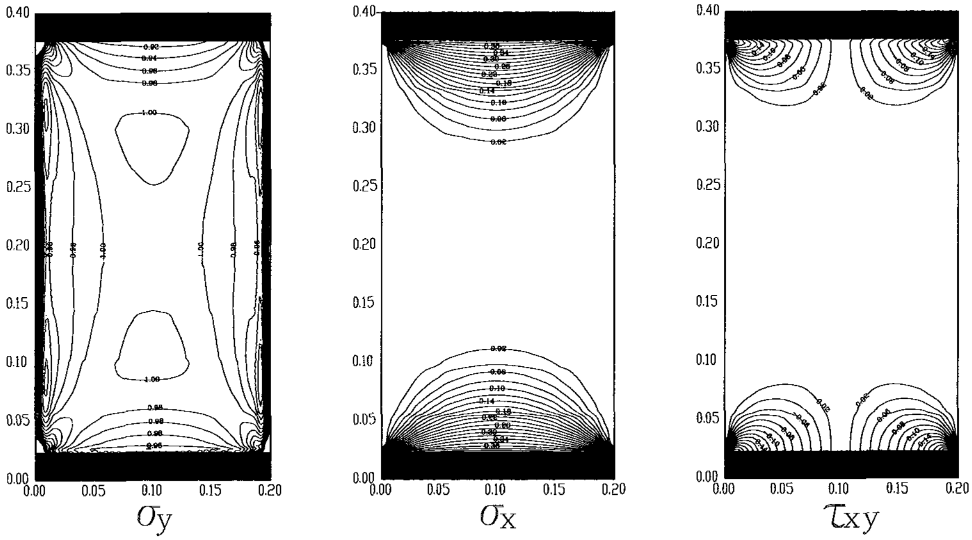

Fig. (11) Stress contour of specimen under uniform compression (Poisson's ratio, ν=0.5, with friction)