A DISCRETE LIMIT ANALYSIS OF FRAMED STRUCTURES INCLUDING THE EFFECTS OF FOUNDATIONS BY USING NEW BEAM ELEMENTS

by

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INTRODUCTION

A family of new discrete elements especially suitable for limit analysis of solids and structures have been derived by the second authors in 1976。[1)∿5)] A new beam element was derived by specializing his general three dimensional element。 Despite of its simplicity, this beam element for bending can take into account of effect of the shear deformation. Its effectiveness was verified through a series of numerical studies. Recently TOI[6)] and WARANABE[7)] pointed out that this model is identical to the finite element proposed by Hughes[8)] et al. if the integration center point is taken at the middle point of the span.

In practice piles or sheet piles are often idealized as beam structures on the elasto-plastic foundations. In the finite element analysis of such structures, it is necessary to derive a new beam stiffness matrix in which effects of the foundations is considered. For this purpose the following 4th order differential equation for beam bending should be solved under appropriate boundary conditions:

$$ETV'''' + kv = 0$$

where v implies the beam displacement. Consequently the resulting stiffness matrix of a beam element becomes complicated even in case of the elastic foundations. And in case of inelastic foundation it is extremely difficult to obtain the corresponding stiffness matrix in practical form. In case of a new beam element, effects of the foundation can be easily represented by a spring system which supports the beam structure continuously[9),10)] and obeys the assumed stress-strain relation (elastic or inelastic).

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And therefore analysis of framed structures on or in the foundations can be successfully made with reasonable computer time and cost.

In this paper accuracy of the elastic solutions by means of the new beam element is examined first and then validity of the proposed method will be demonstrated by solving some numerical examples. Futhermore new discrete limit analysis for framed structures including the effects of elasto-plastic foundations will be discussed.

FORMULATION OF THE NEW BEAM ELEMENT

For simplicity, consider two dimensional framed structures as shown in Fig. 1. Rigid displacement field is assumed in each element, whose nodal displacements are given by the displacement $(\bar{u}, \bar{v}, \bar{\theta})$ of the centroid as shown in Fig. 1. The superscript - indicates local coordinate system.

Spring constants shown in this figure are denoted by (k_N, k_S, k_M) which resist axial force, shear force and bending moment respectively on the contact surface between element I, and element II, the stiffness matrix of which is given by eq.(1).

$$\mathbf{D} = \begin{bmatrix} k_N & 0 & 0 \\ 0 & k_S & 0 \\ 0 & 0 & k_M \end{bmatrix} \qquad \begin{aligned} k_N &= 2EA/\ell \\ k_S &= 2\alpha GA/\ell \\ k_M &= 2EI/\ell \quad (\ell = \ell_1 + \ell_2) \end{aligned} \tag{1}$$

where, ℓ_1 and ℓ_2 are length of individual beam elements respectively, EI bending stiffness, GA shear stiffness, EA axial stiffness and α is the effective shear modulus. The strain energy expression of this new beam element V can be given by the following equation:

$$\begin{aligned} V &= V_N + V_S + V_M \\ V_N &= k_N \delta_N^2/2 \\ V_S &= k_S \delta_S^2/2 \\ V_M &= k_M \varphi^2/2 \end{aligned} \tag{2}$$

where δ_N, δ_S are relative displacements due to axial and shear forces respectively, and φ is relative rotation due to bending moment. These relative displacement vectors $\{\delta\}$ can be given by the following matrix equation:

$$\{\delta\} = [B]\{\bar{u}_i\}$$

$$[B] = \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & \ell_1/2 & 0 & 1 & -\ell_2/2 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \quad (3)$$

$$\{\delta\}^t = \lfloor \delta_N \ \delta_S \ \varphi \rfloor$$

$$\{\bar{u}_i\}^t = \lfloor \bar{u}_1 \ \bar{v}_1 \ \bar{\theta}_1 ; \ \bar{u}_2 \ \bar{v}_2 \ \bar{\theta}_2 \rfloor$$

And the relationship between local and global coordinate systems can be given by the following equation:

$$\{\bar{u}\} = [T]\{u\}$$

$$[T] = \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4)$$

$\{u\}^t = \lfloor u , v , \theta \rfloor$ (global coordinate system)

$\{\bar{u}\}^t = \lfloor \bar{u} , \bar{v} , \theta \rfloor$ (local coordinate system)

Therefore, substituting eq.(1), eq.(3) and eq.(4) to eq.(2), the strain energy of the new beam element can be obtained as the following matrix equation:

$$V = \frac{1}{2}\{u_i\}^t[T]^t[B]^t[D][B][T]\{u_i\}$$

$$= \frac{1}{2}\{u_i\}^t[K]\{u_i\} \quad (5)$$

$$(\because [K] = [T]^t[B]^t[D][B][T])$$

where $[K]$ is a (6 x 6) symmetric matrix.

Applying Castigliano's theorem to eq.(5), the following stiffness equation can be derived;

$$\frac{\partial V}{\partial u_i} = \{P\} = [K]\{u\} \quad (6)$$

where $\{P\}$ is nodal load vector defined by the following equation:

$$\{P\}^t = \lfloor X_1 \ Y_1 \ M_1 \ ; \ X_2 \ Y_2 \ M_2 \rfloor \quad (7)$$

ANALYSIS OF FRAMED STRUCTURES ON THE ELASTIC FOUNDATIONS

In case of finite element analysis of framed structures on or in elastic foundations it is customary to idealize the effect of foundation as the continuous spring system resisting the relative normal and shear displacements due to bond strength between structures and foundations. Numerical model for a beam element on the elastic foundations is shown in Fig. 2, for which the rigid displacement field is assumed.
Where k_h, k_s are coefficient of horizontal subgrade and coefficient of friction respectively. As the rigid displacement field is assumed, linear distribution of the subgrade reaction can be obtained in each element.
By integration of the subgrade reaction along the beam element, it can be shown that the effect of the subgrade reaction can be represented by three springs attatched to the centroid of the beam whose spring constants are given by the following equations.

$$\left.\begin{aligned} K_h\,\bar{v} &= \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} k_h(\bar{v} + \tfrac{1}{2}\,\bar{x}\,\bar{\theta})d\bar{x} = (k_h\,\ell)\bar{v} \\ K_r\,\bar{\theta} &= \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} k_h(\bar{v} + \tfrac{1}{2}\,\bar{x}\,\bar{\theta})d\bar{x} = (\frac{k_h\ell^3}{12})\bar{\theta} \\ K_s\,\bar{u} &= \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} k_s\bar{u}d\bar{x} = (k_s\ell)\bar{u} \end{aligned}\right\} \tag{8}$$

where

$$\left.\begin{aligned} K_h &= k_h\,\ell \\ K_r &= k_h\,\ell^3/12 \\ K_s &= k_s\,\ell \end{aligned}\right\} \tag{9}$$

Now subgrade reaction $\bar{R}$ can be given by the following matrix equation:

$$[\bar{R}] = -\begin{bmatrix} K_s & 0 & 0 \\ 0 & K_h & 0 \\ 0 & 0 & K_r \end{bmatrix}\begin{Bmatrix} \bar{u} \\ \bar{v} \\ \bar{\theta} \end{Bmatrix} = -[\bar{k}]\{\bar{u}\} \tag{10}$$

Subgrade reaction matrix with respect to the local coordinate system can be transformed to the one with respect to global coordinate system by the following matrix $[T]$:

$$[T] = \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

where β is the angle of coordinate axis as shown in Fig. 3. Accordingly subgrade reaction of global coordinate system $[R]$ can be given by using rigid displacement $\{u\}$ (global coordinate system) at centroid as follows:

$$\begin{aligned} [R] = [T]^t[\bar{R}] &= -[T]^t[\bar{k}]\{\bar{u}\} \\ &= -[T]^t[\bar{k}][T]\{u\} \\ &= -[k]\{u\} \end{aligned} \tag{12}$$

$$(\because\ [k] = [T]^t[\bar{k}][T])$$

Therefore total stiffness equation can be given by following equation:

$$\begin{aligned} Ku &= P + R \\ &= P - ku \end{aligned} \tag{13}$$

Thus the problem will be reduced to solve the following equation:

$$(K + k)\ u = P \tag{14}$$

In what follows verification of the proposed method will be demonstrated by solving some numerical exampls. As the first example, consider a pile in the homogeneous elastic foundations such as shown in Fig. 4. Material properties assumed for computation are shown in Fig. 4. Fig. 5 ∿ Fig. 10 show the comparison of the present numerical and analytical solutions. Convergency of the calculated bending moment at arbitrary depth is shown in the (a) which are on the left side of figure. It was found that sufficiently accurate numerical results can be obtained if a pile is divided into 10 elements. Comparison of present numerical and analytical solutions for displacement is shown in the (b) on the center of figure. Even in case of coarse mesh division, accurate result can be obtained. Calculated bending moment distribution in the case of 16 elements division is shown in the (c) which are on the right hand side of figure.

Table 1 shows comparison of present numerical and analytical solutions for displacement at the top of a given pile. Since the effect of shear deformation is taken into account in this model, displacement at the top of a given pile is larger to compare with the analytical solution in case of horizontal loading.

ANALYSIS OF FRAMED STRUCTURES ON THE ELASTO-PLASTIC FOUNDATIONS

A method of new discrete limit analysis for framed structures considering failure of foundations and beam elements will be discribed in this section. Yield condition of framed structures will be discussed first. Choose a yield condition for the bending moment and axial force given by the following equation:

$$\pm\left(\frac{M}{M_{px}}\right) + \left(\frac{p}{p_y}\right)^2 = 1 \tag{15}$$

where M_{px} and p_y is plastic bending moment and yield axial force respectively. Applying flow rule to this yield function, the constitutive equation after yielding can be obtained as follows:

$$[D^p] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ & k_{22} & k_{23} \\ \text{SYM.} & & k_{33} \end{bmatrix} \tag{16}$$

$$k_{11} = k_N - \frac{1}{F}\left(\frac{2p}{p_y^2}\right)^2 k_N$$

$$k_{22} = k_S \qquad k_{12} = k_{23} = 0$$

$$k_{33} = k_M - \frac{1}{F}\left(\frac{1}{M_{px}}\right)^2 k_M$$

$$k_{13} = \mp\left(\frac{2p}{p_y^2}\right)\left(\frac{1}{M_{px}}\right) k_N\, k_M / F$$

$$F = \left(\frac{2p}{p_y^2}\right)^2 k_N + \left(\frac{1}{M_{px}}\right)^2 k_M$$

Nextly the failure of foundations will be considered. Assume the foundations to be linear elastic up to the ultimate strength of soil pressure. After failure of the its, subgrade reaction is considered to be constant and coefficient of horizontal subgrade is assumed to be zero. Analytical solutions can be obtained by this method only in case of homogeneous foundations irrespective of the depth. However, numerical method where new beam elements are employed on or in the elasto-plastic foundations is free from such limitation. And failure of foundations and framed structures can be easily taken into account at the same time by this numerical method. In this paper the ultimate resisting earth pressure for a cohesive soil is assumed as shown in Fig. 11.

(1) Analysis of a pile in the elasto-plastic foundations

Consider a pile as shown in Fig. 12. Assumed material properties of a pile are shown in the figure, and constant of foundations are shown in Fig. 11. A pile with the tip-free condition are divided into elements whose length is 0.5m. In this numerical example a horizontal load is applied to the pile, and ultimate horizontal load and collapse mode are obtained by the present method which are shown in Fig. 13. Numbers in the figure implies the sequence of collapse. Numerical result in case of horizontal load at same height and 3m height above the ground surface are illustrated in upper and lower part of the figure respectively. In case of a short pile, collapse of the pile does not occur and ultimate horizontal load are defined by the failure of foundations. On the other hand in case of a longer pile, plastic hinge is formed and ultimate horizontal load is defined by the failure of foundations and formation of the plastic hinge in a pile.

Numerical results of the displacement are shown in Fig. 14. In case of a shorter pile possibility of tumbling is observed. On the other hand in case of a longer pile the displacement at the top of a pile is found to be smaller.

The relation between the ultimate horizontal load and ratio of pile are shown in Fig. 15. Good agreement between calculated and Broms's limit loads was obtained. In case of a short pile in which plastic hinge is not generated the higher limit load was obtained the pile length becomes longer. However, when the ratio of pile exceeds a certain limit, a plastic hinge is generated in the pile and limit load was found to be constant.

The relation between horizontal displacement at the top of the pile and ratio of pile are shown in Fig. 16. Before generating plastic hinge in the pile very large displacement is observed due to failure of foundations. On the other hand in case where a plastic hinge is formed in the pile, the displacement was found to be constant irrespective of ratio of pile.

(2) Analysis of a portal frame with piles

Next analyzed a portal frame which consist of the same materials as shown in Fig. 17. The piles are divided into element size of 0.5m length and portal frame on the ground are divided into elements of 1m length.

Collapse mode obtained by the present method are shown in Fig. 18. Number in the figure implies the sequence of collapse as shown in example (1). In case of the fixed-support condition, support region broke at first and then yielding of the corner generated. On the other hand in case of portal frame with piles, collapse mechanism is formed by the failure of foundations and piles.

Displacement at final step of calculations are shown in Fig. 19. In case of the fixed-support condition generally small displacement are obtained comparing with another case, because effect of foundations is not taken into account.

The relation between limit load and ration of pile are shown in Fig. 20. And graphes for displacements are shown in Fig. 21. Similar numerical results of a pile were obtained. The bending moment distribution is shown in Fig. 22. In case of the portal frame with piles, bending moment distribution is quite different from those of the structures with fixed support.

CONCLUSION

From the results of some numerical examples illustrated in this paper the following conclusion can be drawn:

(1) In case of the elasto-plastic foundations, behavior of the overall structures is quite different from those of the structure with fixed support.

(2) Location of the maximum bending moment calculated is different from the case of fixed support.

(3) Due to possible failure of the foundation, unexpected larger displacement of framed structures will be observed.
(4) Calculated limit load is smaller than that of the fixed support.
(5) It is impractical to use piles longer than the minimum required length in case of horizontal loading.

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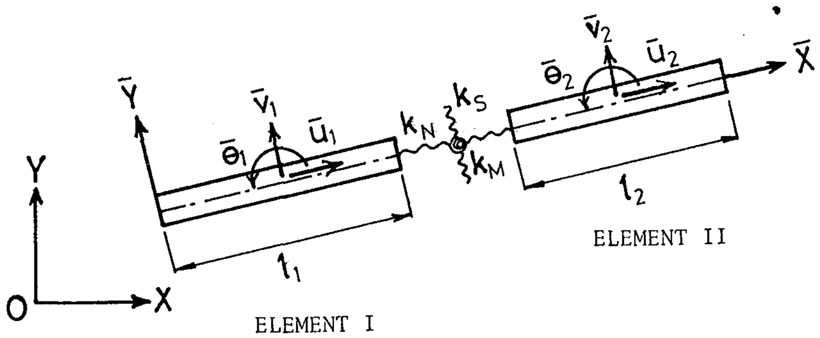


Fig. 1 New discrete beam element

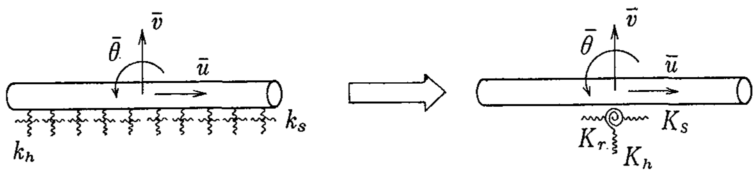


Fig. 2 Constant of subgrade reaction for the new beam element on the elastic foundations

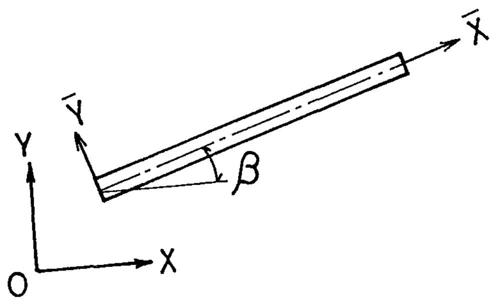
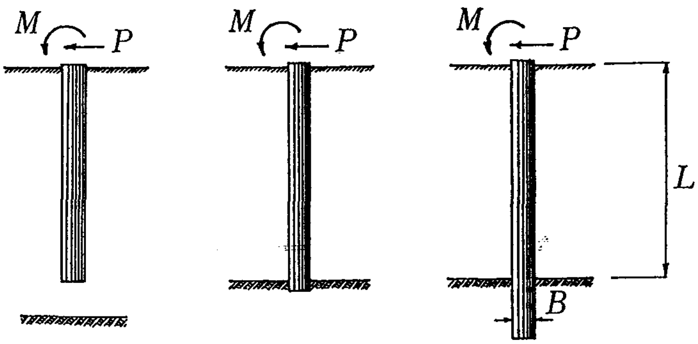


Fig. 3 Local and global coordinate systems



Tip-Free Tip-Hinged Tip-Clamped

Material Properties

Modulus of deformation	350 000 kg/cm^2
Coefficient of horizontal subgrade	0. 5 kg/cm^3
Moment of inertia	103 200 cm^4
Diameter (*B*)	40 cm
Pile length (*L*)	1 000 cm
Shear force (*P*)	2 000 kg
Bending moment (*M*)	300 000 kg·cm

Fig. 4 Numerical model for a beam element on the elastic foundations

Table 1 Displacement for y direction at x = 0

	TIP-FREE		TIP-HINGED		TIP-CLAMPED		
	EXACT	NUMERICAL	EXACT	NUMERICAL	EXACT	NUMERICAL	No.
P=2.t	0.6869	0.6709	0.6856	0.6756	0.6812	0.6725	4
		0.6871		0.6858		0.6818	8
		0.6877		0.6864		0.6821	16
M=3.tm	0.3532	0.3087	0.3538	0.3093	0.3503	0.3065	4
		0.3423		0.3429		0.3396	8
		0.3505		0.3511		0.3476	16

No. implies the numbers of elements used in the present numerical analysis

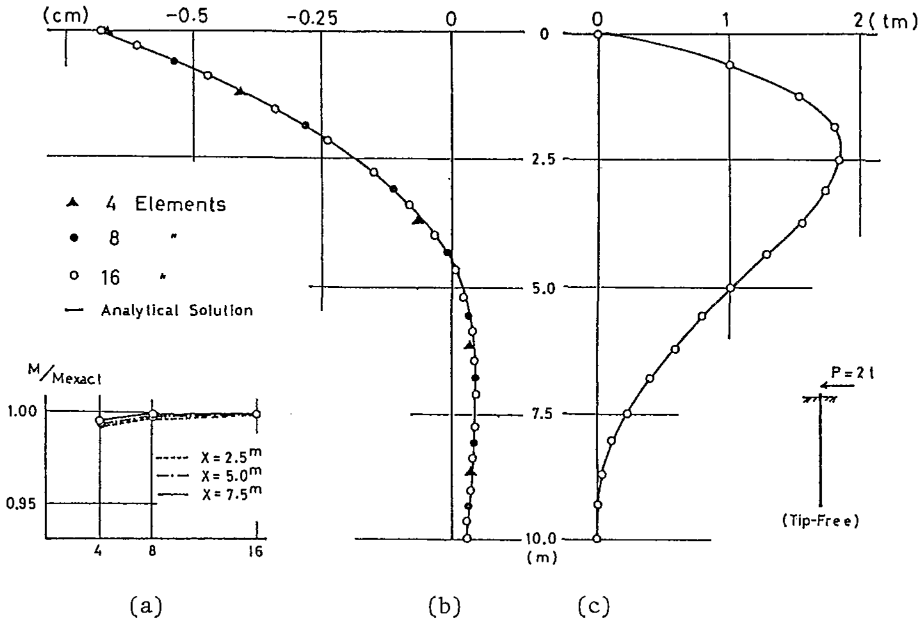


Fig. 5 Deflected shape and bending moment distribution of a pile with the tip-free condition

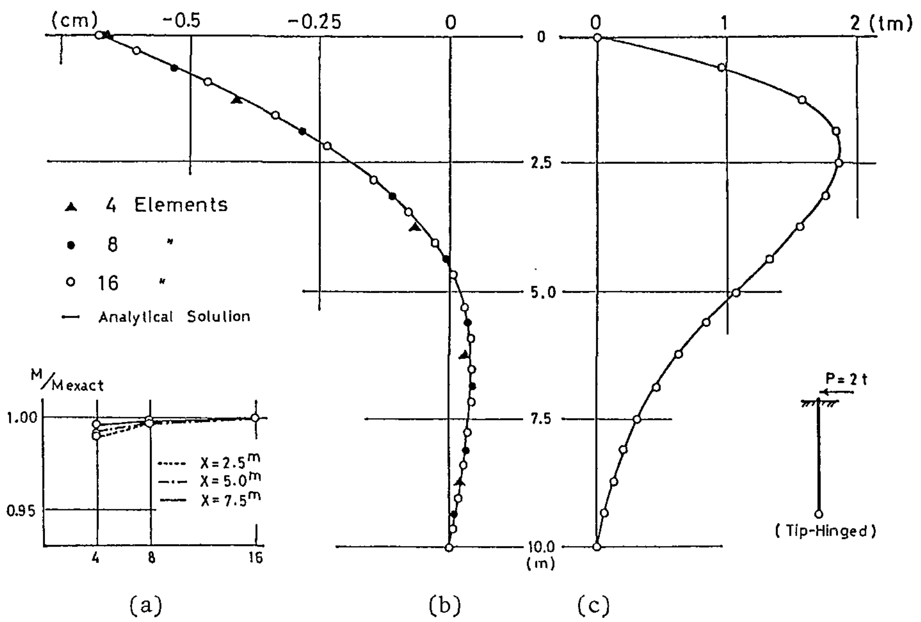


Fig. 6 Deflected shape and bending moment distribution of a pile with tip-hinged condition

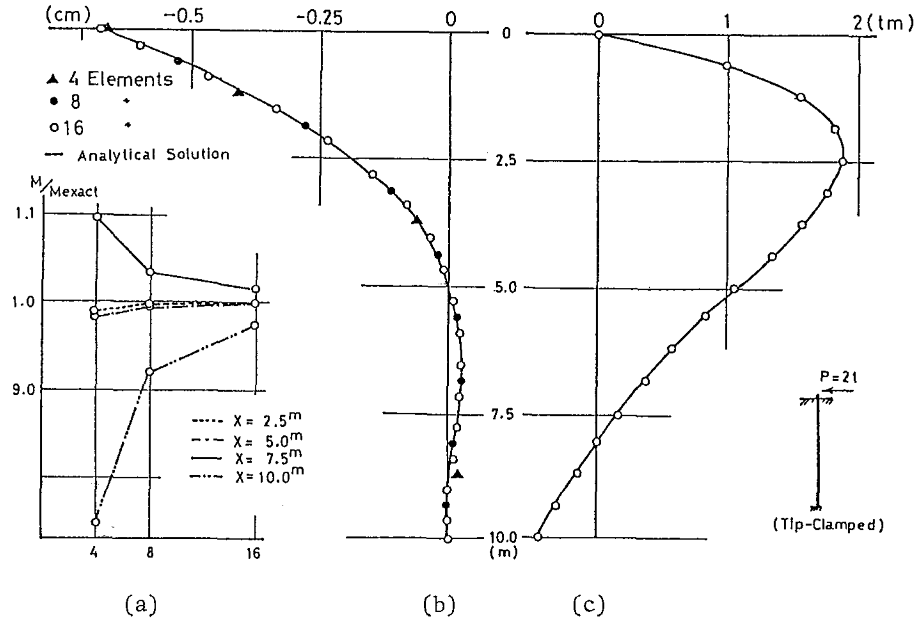


Fig. 7 Deflected shape and bending moment distribution of a pile with tip-clamped condition

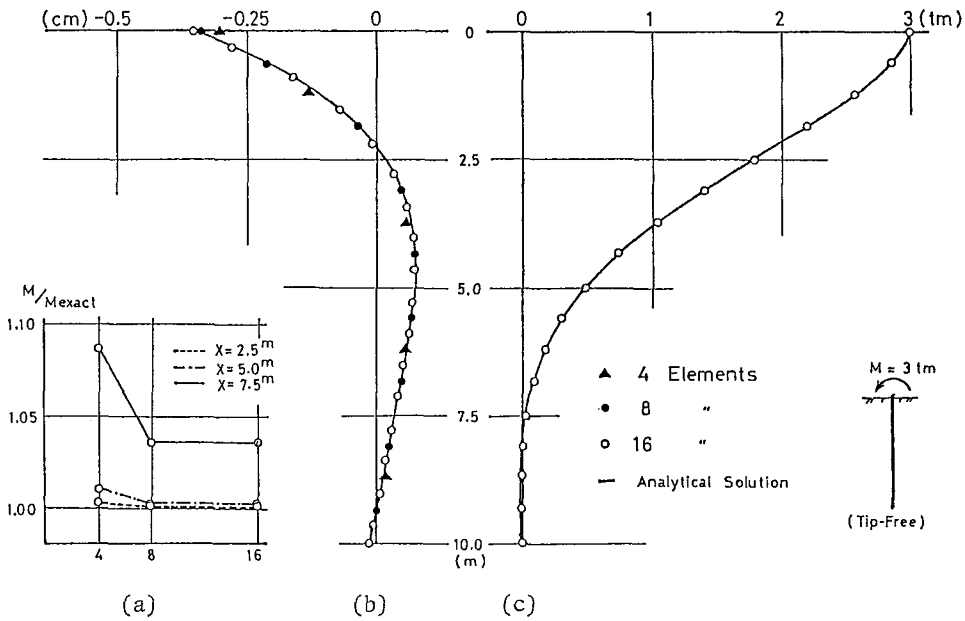


Fig. 8 Deflected shape and bending moment distribution of a pile with tip-free condition

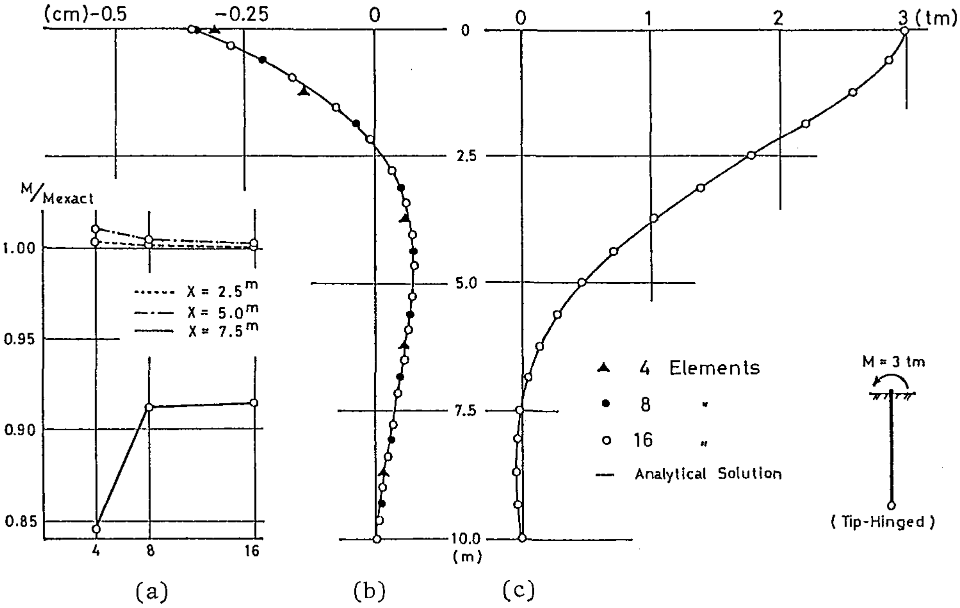


Fig. 9 Deflected shape and bending moment distribution of a pile with tip-hinged condition

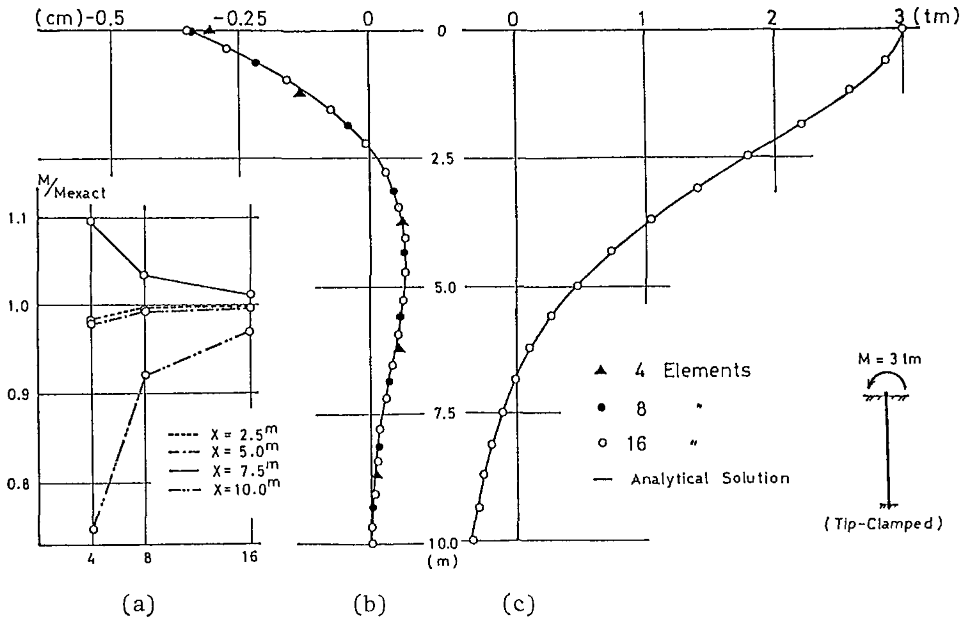


Fig. 10 Deflected shape and bending moment distribution of a pile with tip-clamped condition

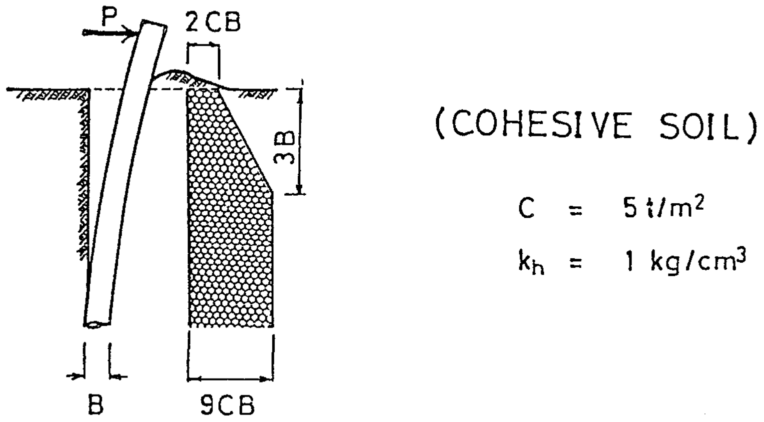


Fig. 11 Ultimate resisting earth presure for a cohesive soil

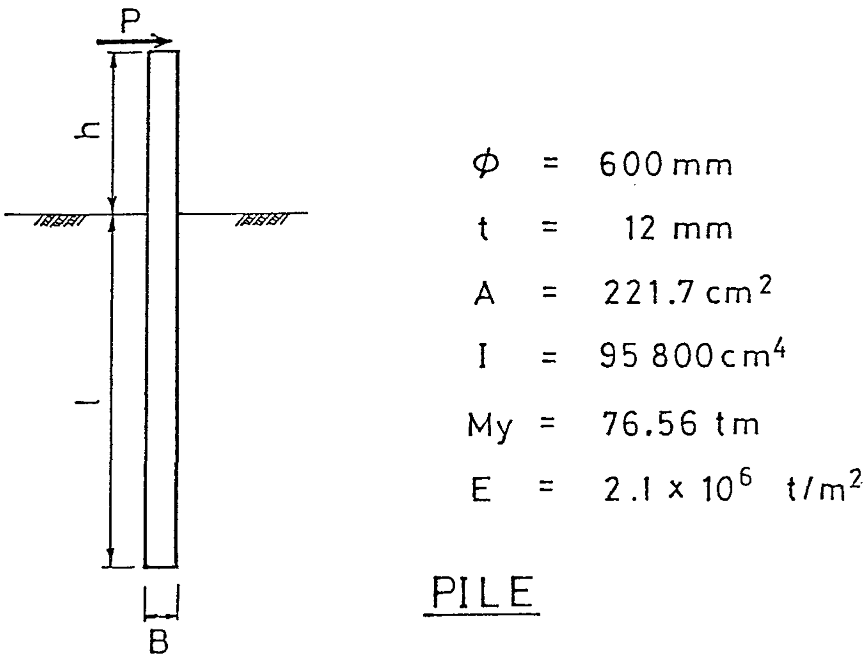


Fig. 12 Numerical model for a pile on the elasto-plastic foundations

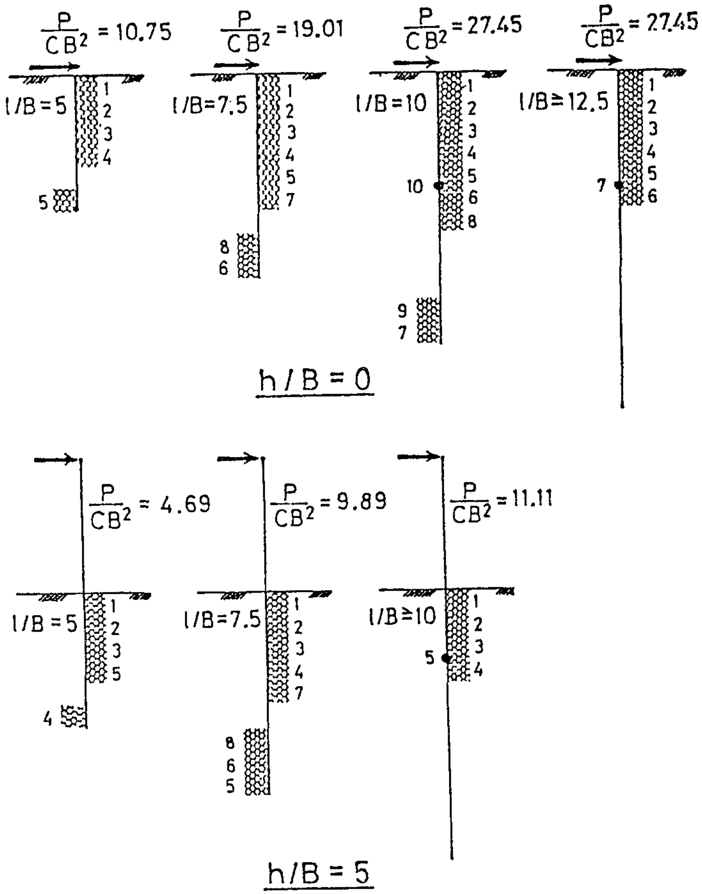


Fig. 13 Ultimate horizontal load and collapse mode obtained by the present method

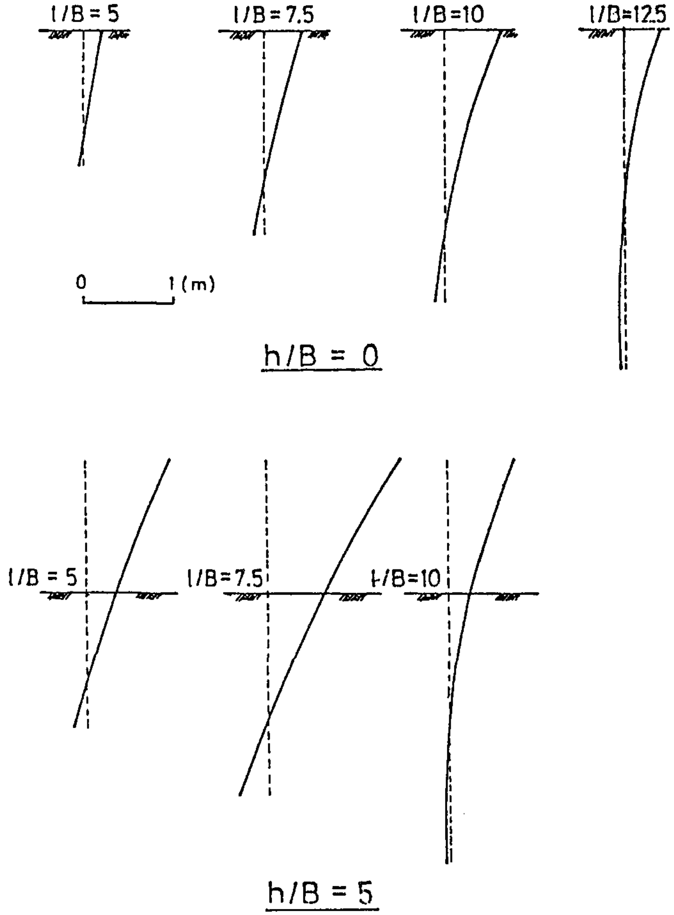


Fig. 14 Deflected shape of the pile

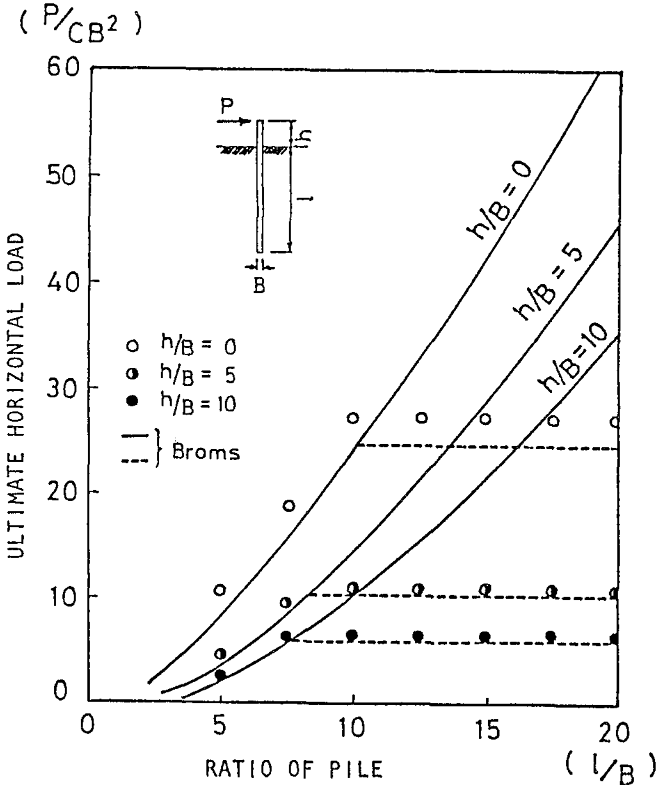


Fig. 15 Comparison of the present numerical and analytical solution for the limit load of a pile

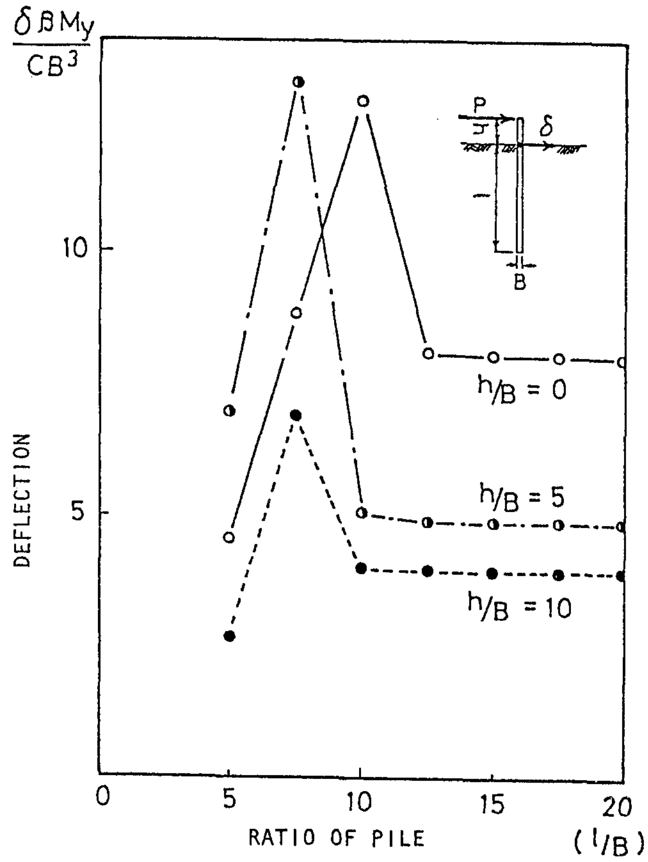


Fig. 16 Result of calculation for the displacement of a pile

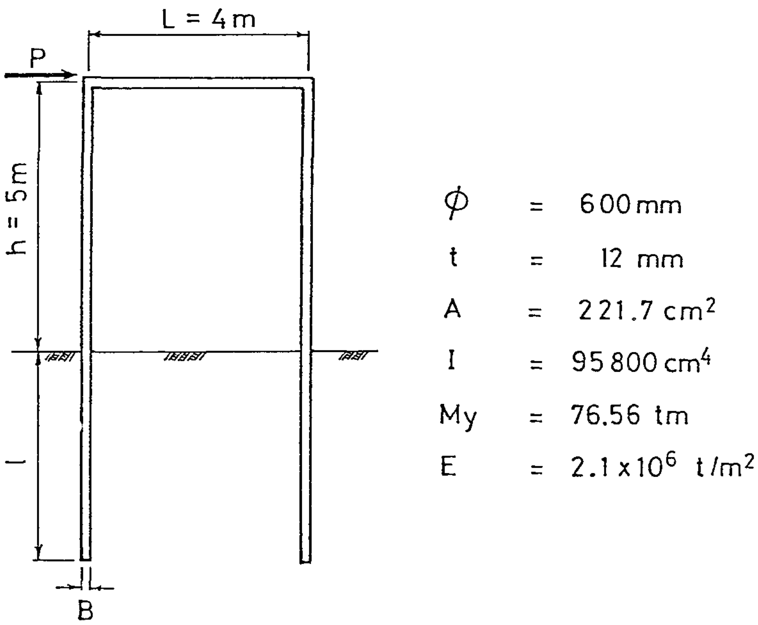


Fig. 17 Numerical model for the portal frame with piles

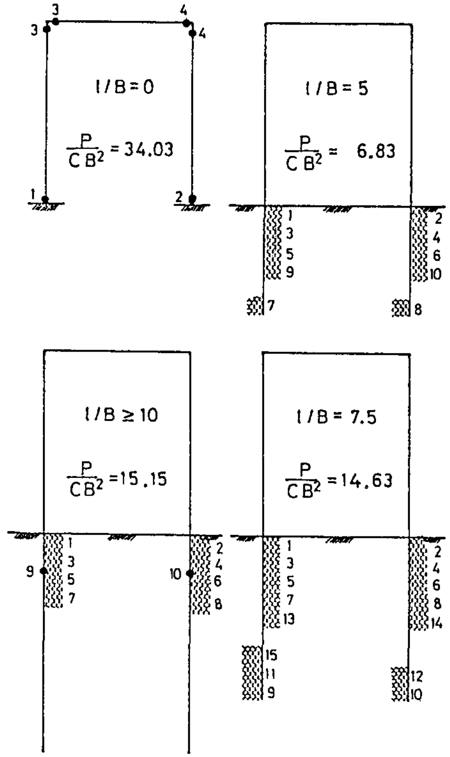


Fig. 18 Limit load and collapse mode obtained by the present method

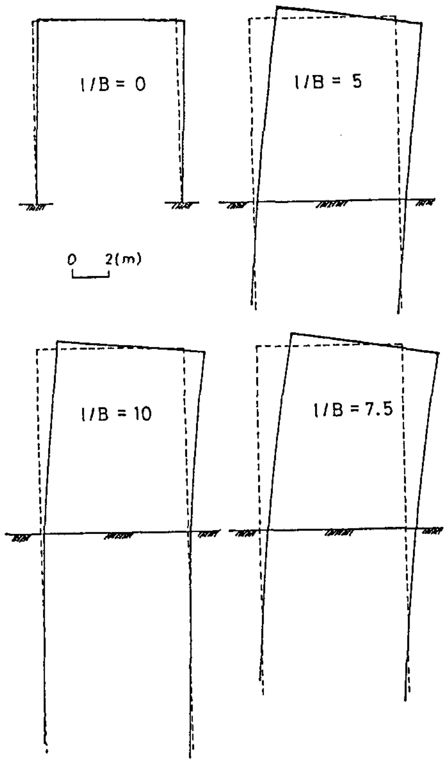


Fig. 19 Deflected shape of the portal frame

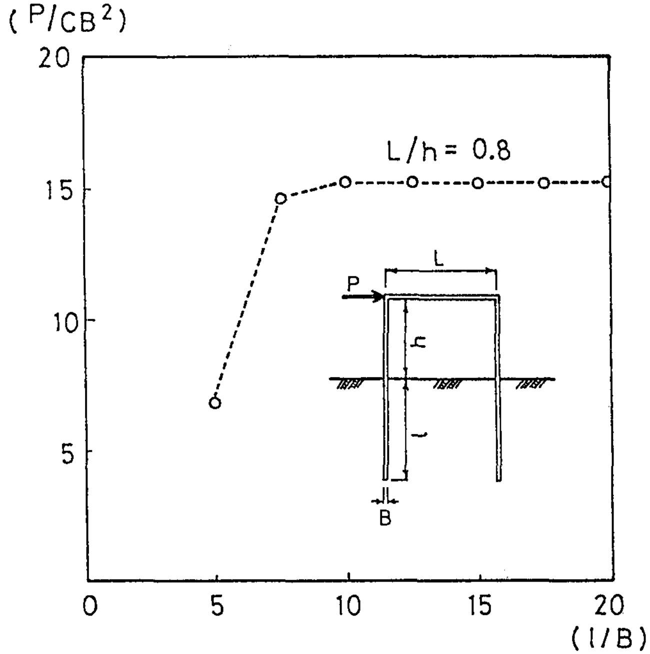


Fig. 20 Ultimate horizontal load for the portal frame with piles

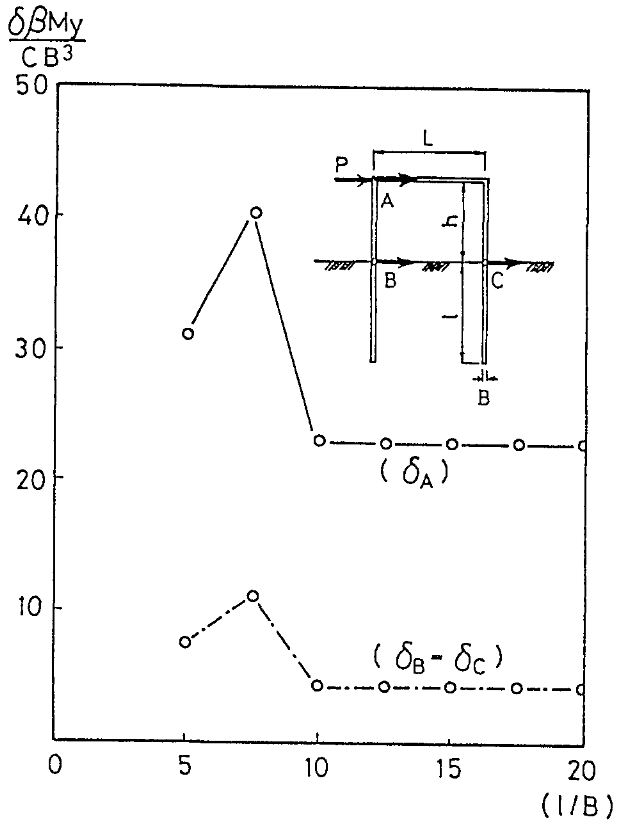


Fig. 21 Result of calculation for the horizontal displacement

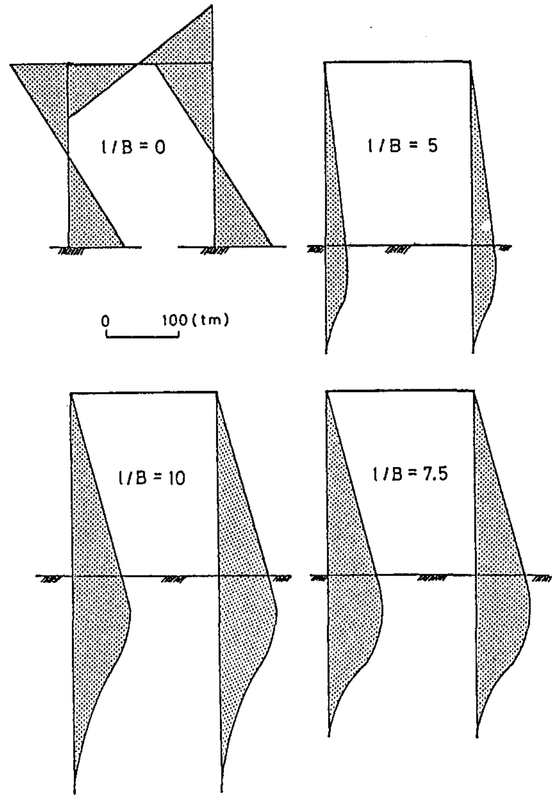


Fig. 22 Bending moment distribution of the portal frame with piles