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COMPARISON OF FIVE APPROXIMATE METHODS OF THE NONLINEAR EQUATION OF MOTION

by

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SYNOPSIS

For the analysis of the nonlinear equation of motion, we have many analytical methods and are often puzzled to select the most suitable method for the problem to be analyzed. In this paper, in order to examine the properties of several approximate methods, we present the results of the first and second order approximations of each method for the one-degree-of-freedom system, which contains both the quadratic and cubic terms. Then we clarify the differences between each approximate method by comparing the backbone curves of the first characteristic mode of a sinusoidal arch.

INTRODUCTION

Shallow structures like arches and spherical shells under static loadings have the load-deflection curves as shown in Fig. 1. Governing equations of these curves contain quadratic terms, together with cubic ones. These quadratic terms make the nonlinear behavior unsymmetrical and cause the snap-through and direct snapping phenomena in the static and dynamic buckling problems, respectively. The first step for treating the stability of shallow structures is to examine the nonlinear characteristic like the backbone curve of the nonlinear equations of motion with quadratic and cubic terms.

We have many analytical methods and are often puzzled to select the suitable method for the problem to be analyzed. On the other hand, a rapid development of the finite element method in the nonlinear field has enabled us to solve large scale algebraic simultaneous nonlinear equations.

Under these circumstances, it is the purpose of the present paper to compare five kinds of analytical methods by numerically constructing backbone curves for the one-degree-of-freedom system.

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COMPILATION OF FIRST AND SECOND ORDER APPROXIMATIONS

Let us set a typical equation of one-degree-of-freedom as

$$\ddot{x} + ax + bx^2 + cx^3 = 0 \tag{1}$$

When b=0, the equation (1) is called as Duffing's type. Since the coefficient b plays an important role, as aforesaid, in shell-like structures, let us examine the influence of b by comparing with the case it vanishes.

In this section, the results of both the first and second order approximations obtained by each method are compiled, without the procedure of derivation. In the following, x_I and x_{II} represent the first and second order approximation, respectively.

[1] Averaging method

$$x_I = A\cos\omega_I t$$

$$\omega_I = (1 + 3cA^2/8a)\omega_o$$

$$x_{II} = A\cos\omega_{II}t + b\{A^2/4\omega_o{}^2 + A^2/12\omega_o{}^2\cdot(4\cos\omega_{II}t - 7\cos2\omega_{II}t) + cA^3/32\omega_o{}^2\cdot(\cos3\omega_{II}t - \cos\omega_{II}t)$$

$$\omega_{II} = (1 + 3cA^2/8a)\omega_o$$

[2] Bogoliuboff-Mitropolsky's asymptotic method

$$x_I = A\cos\phi + b(-A^2/2a + A^2/6a\cdot\cos2\phi) + cA^3/32a\cdot\cos3\phi$$

$$\phi = \omega_I t = (1 + 3cA^2/8a)\omega_o t$$

$$x_{II} = A\cos\phi + (-bA^2/2a + 5bcA^4/8a^2) + (bA^2/6a - 31bcA^4/96a^2)\cos2\phi + (cA^3/32a - 21c^2A^5/1024 + b^2A^3/48a^2)\cos3\phi + bcA^4/96a^2\cdot\cos4\phi + c^2A^5/1024a^2\cdot\cos5\phi$$

$$\phi = \omega_{II}t = (1 + 3cA^2/8a - 5b^2A^2/12a^2 - 15c^2A^4/256a^2)\omega_o t$$

[3] Perturbation method (1)
(expansion in the neighborhood of ω_o)

$$x_I = A\cos\phi - bA^2/2a + (b/3a - cA/32a)A^2\cos\phi + bA^2/6a\cdot\cos2\phi + cA^3/32a\cdot\cos3\phi$$

$$\phi = (1 + 3cA^2/8a)\omega_o t$$

$$x_{II} = x_I + (-b^2A^3/3a^2 + 21bcA^4/32a^2) + (87b^2A^3/432a^2 - 35bcA^4/96a^2 + 23c^2A^5/1024a^2)\cos\phi + (b^2A^3/9a^2 -$$

$bcA^4/3a^2)\cos 2\phi + (-3c^2A^5/128a^2 + b^2A^3/48a^2 + bcA^4/32a^2)\cos 3\phi + bcA^4/96a^2 \cdot \cos 4\phi + c^2A^5/1024a^2 \cdot \cos 5\phi$

$$\phi = (1 + 3cA^2/8a - 5b^2A^2/12a^2 + bcA^3/4a^2 - 21c^2A^4/256a^2)\omega_o t$$

[4] Perturbation method (2)
(expansion in the neighborhood of ω)

$$x_I = A\cos\phi - bA^2/2a + (bA^2/3a - cA^3/32a)\cos\phi + bA^2/6a \cdot \cos 2\phi + cA^3/32a \cdot \cos 3\phi$$

$$\phi = (1 + 3cA^2/8a)\omega_o t$$

$$x_{II} = x_I + (-b^2A^3/3a^2 + 9bcA^4/8a^2) + (29b^2A^3/144a^2 - 23bcA^4/24a^2 + 11c^2A^5/1024a^2)\cos\phi + (b^2A^3/9a^2 - 5bcA^4/24a^2)\cos 2\phi + (-3c^2A^5/256a^2 + b^2A^3/48a^2 + bcA^4/32a^2)\cos 3\phi + bcA^4/96a^2 \cdot \cos 4\phi + c^2A^5/1024a^2 \cdot \cos 5\phi$$

$$\phi = (1 + 3cA^2/8a - 5b^2A^2/12a^2 + bcA^3/4a^2 - 21c^2A^4/256a^2)\omega_o t$$

[5] Perturbation method (3)
(expansion in the neighborhood of ω_o^2)

$$x_I = A\cos\phi - bA^2/2a + (bA^2/3a - cA^3/32a)\cos\phi + bA^2/6a \cdot \cos 2\phi + cA^3/32a \cdot \cos 3\phi$$

$$\omega_I^2 = (1 + 3cA^2/4a)\omega_o^2$$

$$x_{II} = x_I + (-bA^3/3a^2 + 21bcA^4/32a^2) + (29b^2A^3/144a^2 - 4bcA^4/3a^2 + 117c^2A^5/1024a^2 - b^2A^4/9216a^2 + 2bcA^5/98304a^2 - c^2A^6/1024^2a^4)\cos\phi + (b^2A^3/9a^2 - bcA^4/3a^2)\cos 2\phi + (-117c^2A^5/1024a^2 + b^2A^3/48a^2 + bcA^4/a^2)\cos 3\phi + bcA^4/96a^2 \cdot \cos 4\phi + (b^2A^4/9216a^2 - bcA^4/49152a^2 + c^2A^6/1024^2a^4)\cos 5\phi$$

$$\omega_{II}^2 = (1 + 3cA^2/4a + bcA^3/a^2 - 5b^2A^2/6a^2 - 9c^2A^4/128a^2)\omega_o^2$$

[6] Perturbation method (4)
(expansion in the neighborhood of ω^2)

$$x_I = A\cos\phi - bA^2/2a + (bA^2/3a - cA^3/32a)\cos\phi + bA^2/6 \cdot \cos 2\phi + cA^3/32a \cdot \cos 3\phi$$

$$\omega_I^2 = (1 + 3cA^2/4a)\omega_o^2$$

$$x_{II} = x_I + (-b^2A^3/3a^2 + 9bcA^4/32a^2) + (29b^2A^3/144a^2 - 11bcA^4/96a^2 - c^2A^5/1024a^2)\cos\phi + (b^2A^3/9a^2 - 5bcA^4/24a^2)\cos 2\phi + (b^2A^3/48a^2 + bcA^4/32a^2)\cos 3\phi + bcA^4/96a^2\cdot\cos 4\phi + c^2A^5/1024a^2\cdot\cos 5\phi$$

$$\omega_{II}^2 = (1 + 3cA^2/4a - 5b^2A^2/6a^2 + 3c^2A^4/128a^2)\omega_o^2$$

[7] Duffing's iteration method

$$x_I = A\cos\phi - bA^2/2a + bA^2/6a\cdot\cos 2\phi + cA^3/32a\cdot\cos 3\phi$$

$$\omega_I^2 = (1 + 3cA^2/4a)\omega_o^2$$

$$x_{II} = A\cos\phi + 1/a^2\cdot\sum_{\substack{i=0\\(i\neq 1)}}^{9} Hi/(1-i^2)\cdot\cos i\phi$$

$$\omega_{II}^2 = (1 + 3cA^2/4a - 5b^2A^2/6a^2 + 35b^2cA^4/64a^3 + 3c^2A^4/128a^2 + 3c^3A^6/2048a^3 - 11b^2c^2A^6/1536a^4)\omega_o^2$$

where
$H_o = (\omega^2 - a)e_o - bE_o - cF_o$,
$H_2 = (\omega^2 - a)e_2 - bE_2 - cF_2$,
$H_3 = (\omega^2 - a)e_3 - bE_3 - cF_3$,
$H_4 = -bE_4 - cF_4$, $H_5 = -bE_5 - cF_5$,
$H_6 = -bE_6 - cF_6$, $H_7 = -cF_7$, $H_8 = -cF_8$,
$H_9 = -cF_9$,

$E_o = e_o^2 + e_1^2/2 + e_2^2/2 + e_3^2/2$,
$E_1 = e_1^2/2 + 2e_oe_2 + e_1e_3$,
$E_3 = 2e_oe_3 + e_1e_2$, $E_4 = e_2^2/2 + e_1e_3$,
$E_5 = e_2e_3$, $E_6 = e_3^2/2$

$F_o = e_o^3 + 3e_oe_1^2/2 + 3e_oe_2^2/2 + 3e_oe_3^2/2 + 3e_1^2e_2/4 + 3e_1e_2e_3/2$
$F_2 = 3e_2^3/4 + 3e_oe_1^2/2 + 3e_o^2e_2 + 3e_oe_1e_3 + 3e_1^2e_2/2 + 3e_1e_2e_3/2 + 3e_2e_3^2/2$
$F_3 = 3e_3^3/4 + 3e_o^2e_3 + 3e_oe_1e_2 + e_1^3/4 + 3e_1^2e_3/2 + 3e_1e_2^2/4 + 3e_2^2e_3/2$
$F_4 = 3e_oe_2^2/2 + 3e_oe_1e_3 + 3e_1^2e_2/4 + 3e_1e_2e_3/2 + 3e_2e_3^2/4$
$F_5 = 3e_oe_2e_3 + 3e_1e_2^2/4 + 3e_1^2e_3/2$
$F_6 = 3e_oe_3^2/2 + 3e_1e_2e_3/2 + e_2^3/4$
$F_7 = 3e_1e_3^2/4 + 3e_2^2e_3/4$
$F_8 = 3e_2e_3^2/4$, $F_9 = e_3^3/4$

$e_o = -bA^2/2a$, $e_1 = A$, $e_2 = bA^2/6a$, $e_3 = cA^3/32a$

[8] Harmonic balance method

Assuming the displacement x, as follows,

$$x = C_o + C_1 \cos\omega t \tag{2}$$

we obtain the simultaneous nonlinear algebraic equations:

$$\begin{aligned} aC_o + b(C_o^2 + C_1^2/2) + c(C_o^3 + 3C_oC_1^2/2) &= 0 \\ (a - \omega^2)C_1 + 2C_oC_1 \cdot b + c(3C_o^2C_1 + 3C_1^3/4) &= 0 \end{aligned} \tag{3}$$

Solving Eq. (3) numerically, we obtain the first order approximation.

NUMERICAL RESEARCH

The purpose of this section is to examine the characteristics of five analytical methods of the nonlinear equation of motion, comparing the numerical results for the backbone curves.

As an illustrative model for the nonlinear equation of motion, we adopt a shallow sinusoidal arch supported hinges (Fig. 1). Restricting the deformation to the first characteristic mode with no damping, we get the equation of motion with both the quadratic and cubic terms:

$$\ddot{x} + (1 + H^2/2)x - 3H/4 \cdot x^2 + 1/4 \cdot x^3 = 0 \tag{4}$$

where H is the shape parameter ($H = h/\sqrt{I/A}$).
Introducing the nondimensional parameters, τ and ξ, as

$$\tau = \sqrt{1 + H^2/2} \cdot t, \qquad x = 2\sqrt{1 + H^2/2} \cdot \xi \tag{5}$$

we obtain the nondimensionalized equation of motion:

$$\ddot{\xi} + \xi + \varepsilon\xi^2 + \xi^3 = 0 \tag{6}$$

in which $\varepsilon = -3H/2\sqrt{1 + H^2/2}$
In the numerical calculation, we adopt two cases of H=3 and H=8, which correspond to the relatively shallow and deep arch, respectively, from the view point of the static buckling problem.

In order to compare the numerical results from each method, it is necessary to conform the correct solution. However, as it is impossible to derive the rigorous solution of Eq. (3), the results by the harmonic balance method of 10 waves approximation will be used for the comparative study (see Fig. 3). In the numerical calculation by using the harmonic balance method for Eq. (6), it is found that the Fourier series converges very rapidly and sufficient accuracy is obtained by taking only up to the second term.

Fig. 4 through 10 show the results by the asymptotic method, perturbation methods of four types, Duffing's iteration method and harmonic balance method of two waves approximation, respectively. The result by the averaging method is not shown because of its unusual error. From these figures, the following comments will be given.

Every method possesses a relatively good precision in the neighborhood of the linear oscillation, that is $0.95 \leq \omega \leq 1.0$. The equation of motion Eq. (6) with quadratic terms has both the softening and hardening properties. The present results show that the numerical methods which can reflect the above combined properties are only the Duffing's method and the harmonic balance method. And for the case with a large quadratic terms of H=8, only the harmonic balance method catches the nonlinear characteristics of the very large amplitude oscillation. We show the results of the case $\varepsilon=0$ in Eq. (6), in Fig. 11 through Fig. 18 for purposes of reference.

In a concluding remark, the numerical results explain that when we apply the perturbation method, asymptotic method, etc. to the nonlinear equation of motion with both the quadratic and cubic terms, it is necessary to introduce a technique which can express the shift between the hardening and softening properties.

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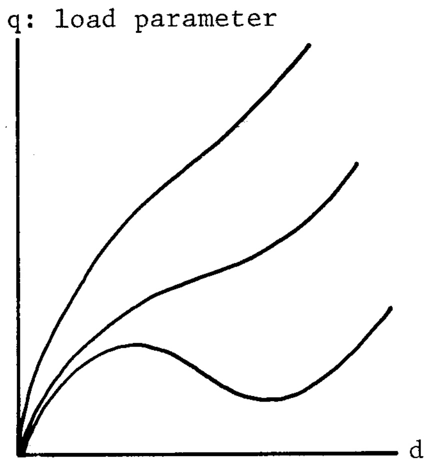

Fig. 1: Typical Load-Deflection Curve

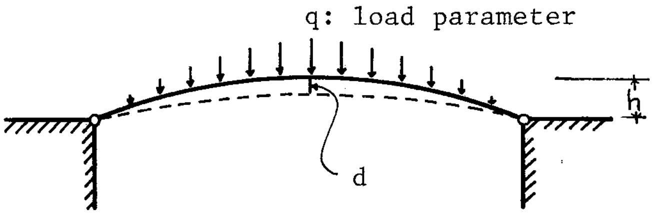

Fig. 2: Shallow Arch

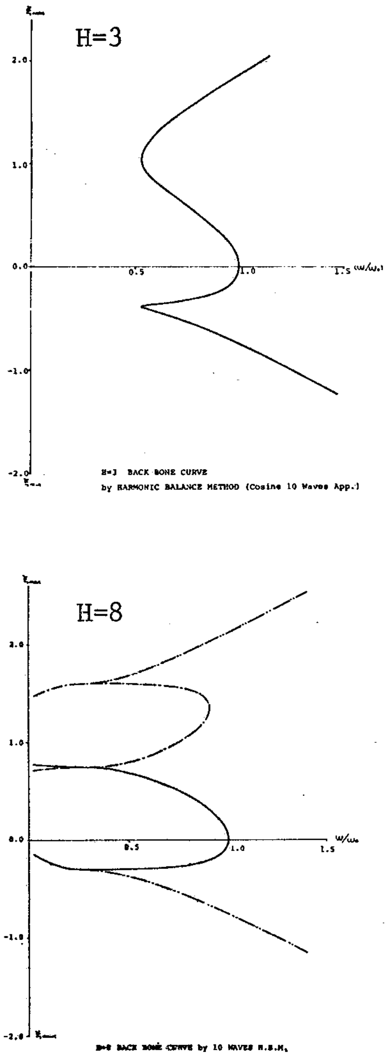

Fig. 3: Backbone Curves
by Harmonic Balance Method
(Cosine 10 Waves App.)

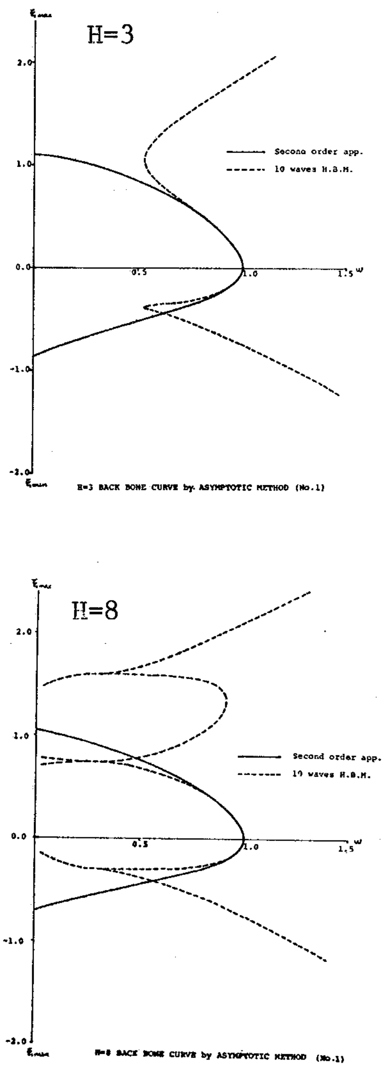

Fig. 4: Backbone Curves
by Bogoliuboff-Mitropolsky's
Asymptotic Method

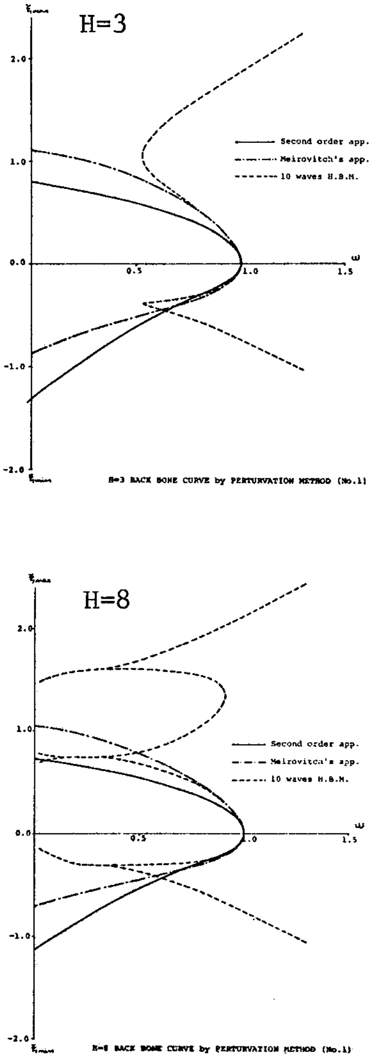

Fig. 5: Backbone Curves by Perturbation Method (No. 1)

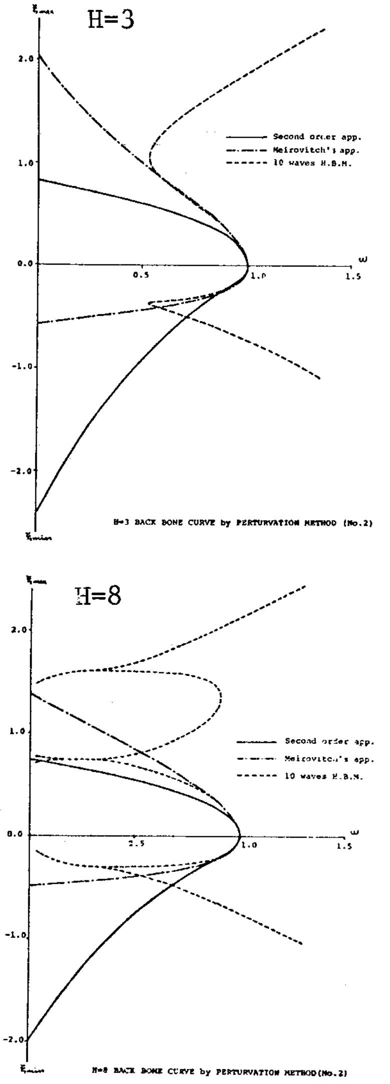

Fig. 6: Backbone Curves by Perturbation Method (No. 2)

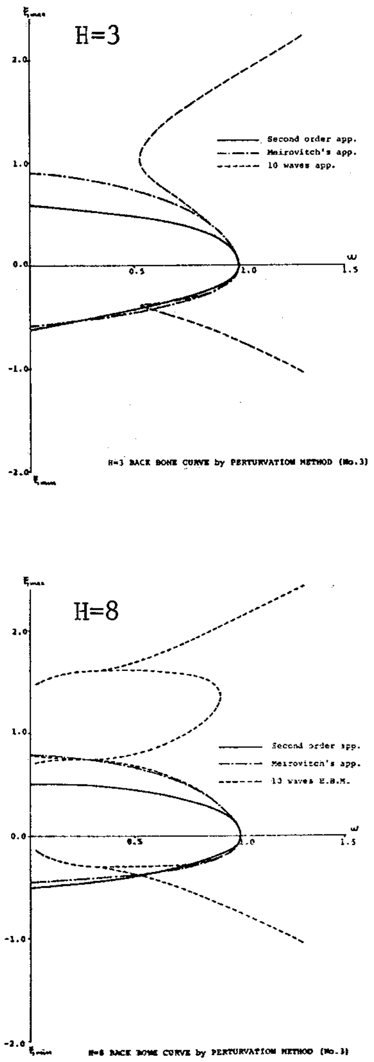

Fig. 7: Backbone Curves by Perturbation Method (No. 3)

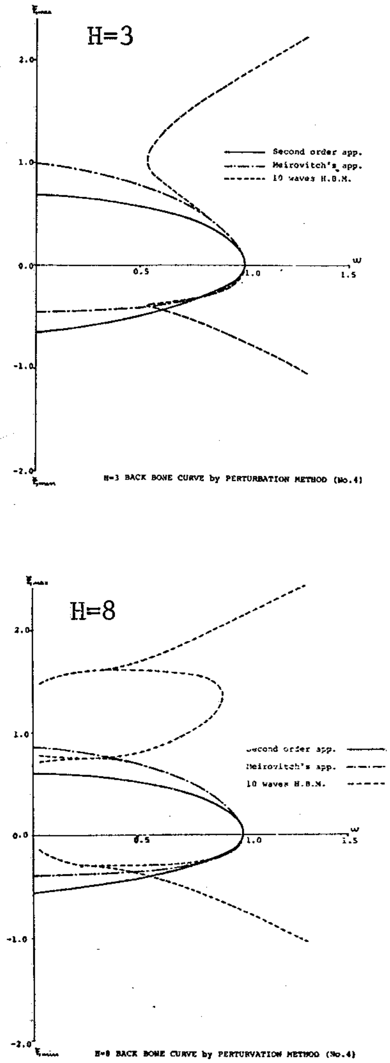

Fig. 8: Backbone Curves by Perturbation Method (No. 4)

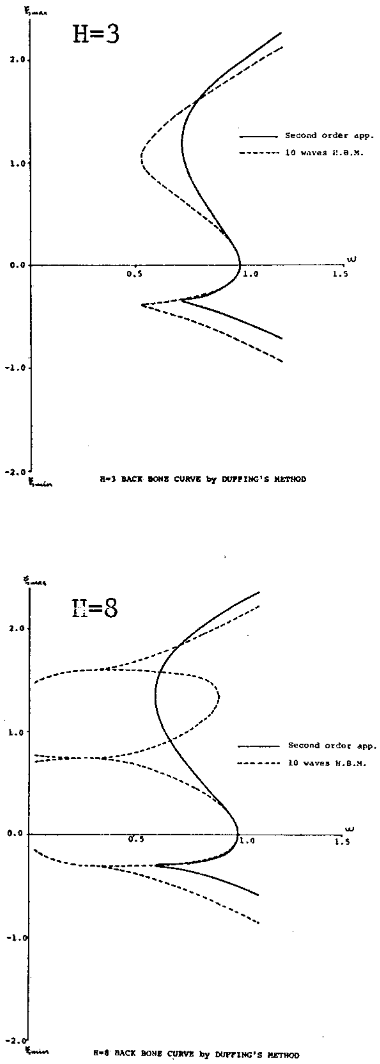

Fig. 9: Backbone Curves by Duffing's Iteration Method

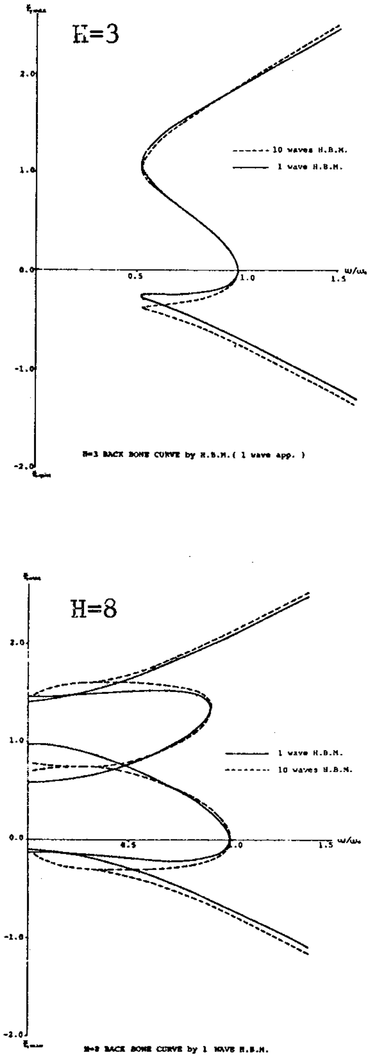

Fig. 10: Backbone Curves by Harmonic Balance Method (Cosine 10 Waves App.)

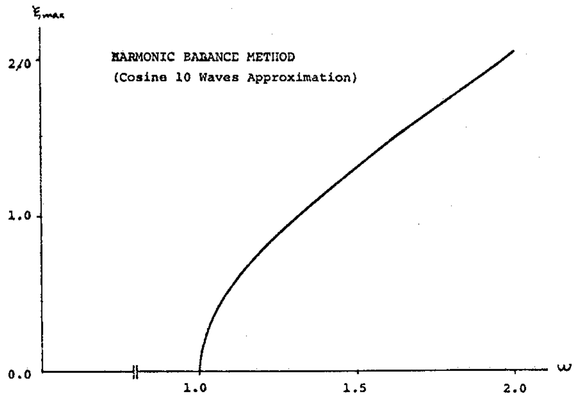

Fig. 11: Backbone Curve by Harmonic Balance Method (Cosine 10 Waves App.)

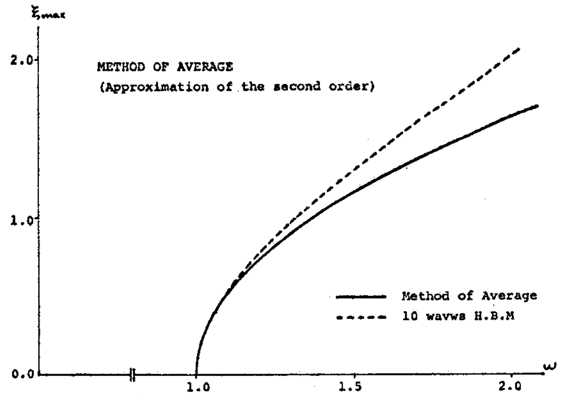

Fig. 12: Backbone Curve by Averaging Method

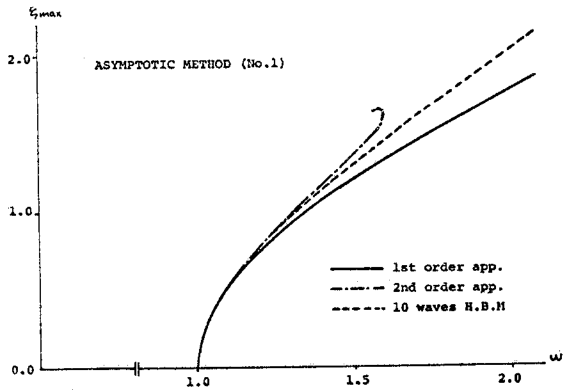

Fig. 13: Backbone Curve by Bogoliuboff-Mitropolsky's Asymptotic Method

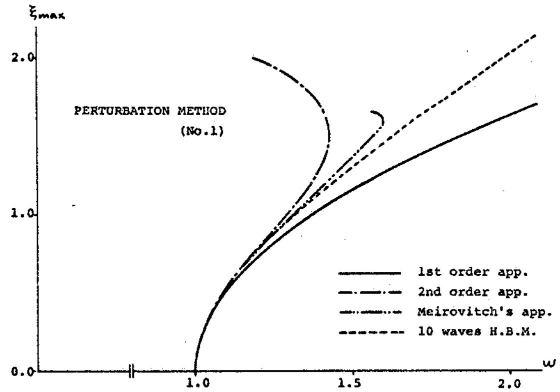

Fig. 14: Backbone Curve by Perturbation Method (No. 1)

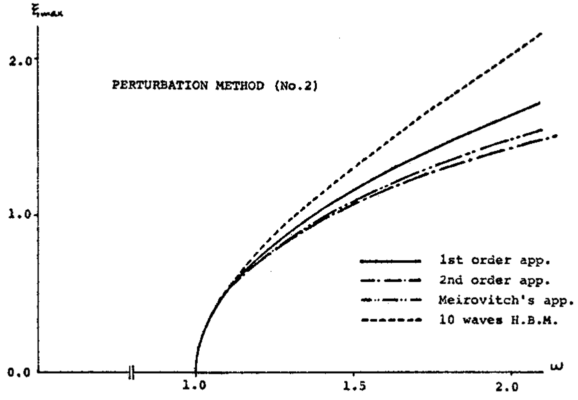

Fig. 15: Backbone Curve by Perturbation Method (No. 2)

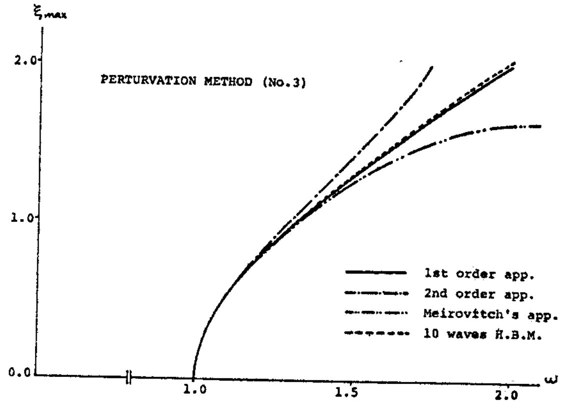

Fig. 16: Backbone Curve by Perturbation Method (No. 3)

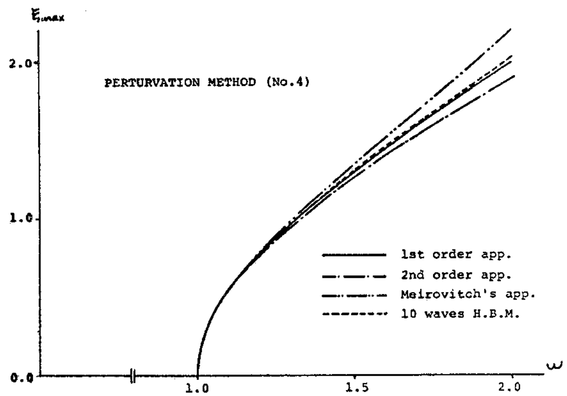

Fig. 17: Backbone Curve by Perturbation Method (No. 4)

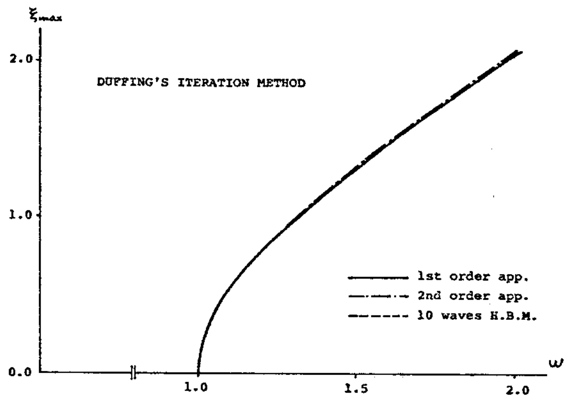

Fig. 18: Backbone Curve by Duffing's Iteration Method