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BRIEF NOTE

STABILITY ANALYSIS OF A CIRCULAR ARCH UNDER HYDROSTATIC PRESSURE BY THE ADJOINT VARIATIONAL METHOD

by

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Introduction

Hydrostatic pressure remains perpendicular to the deformed surface of structure and belongs to the follower-type surface loads. The problem of stability of equilibrium of elastic structures subjected to follower-type surface tractions has recently been approached by several methods (1,2), which contain the adjoint variational method (3).

In this brief note, the adjoint variational method is applied, from the view point of linear eigenvalue problem, to the stability analysis of a circular arch under hydrostatic pressure. Then, numerical results are compared with the buckling loads which have been obtained based on the assumption of the preservation of the initial load direction.

Adjoint Systems and Stationary Value[3,4]

Consider a linear boundary value problem (L,B), where L represents a linear differential operator defined within a domain V and B a boundary condition incidental to L. The system (L,B) is called the "original system."

Corresponding to the original system, we introduce the "adjoint system (L*,B*)" defined by

$$\langle gL^*g^*\rangle = \langle g^*Lg\rangle, \quad \langle\cdots\rangle = \int_v \cdots dv \tag{1}$$

Here g and g^* are any functions satisfying B and B*, respectively. In the special case that the two systems (L,B) and (L*,B*) are coincident, it is said to be "self-adjoint."

Let us define two eigenvalue problems:

$$L\phi_i = \lambda_i\phi_i \quad \mathit{and} \quad L^*\phi_i^* = \lambda_i^*\phi_i^* \tag{2}$$

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where ϕ_i , ϕ_i^* represent the i -th eigenfunctions belonging to each system and λ_i , λ_i^* the eigenvalues. From Eqs. (1) and (2), it follows that

$$\lambda_i \langle \phi_i \phi_j^* \rangle = \langle \phi_j^* L \phi_i \rangle = \langle \phi_i L^* \phi_j^* \rangle = \lambda_j^* \langle \phi_i \phi_j^* \rangle \tag{3}$$

From the above equation, we can show that the sets of eigenvalues λ_i and λ_j^* are identical and the sets ϕ_i and ϕ_j^* are dual, i.e., $\langle \phi_i \phi_j^* \rangle = 0 (i \neq j)$. Moreover, we have

$$\lambda_i = \lambda_i^* = \frac{\langle \phi_i^* L \phi_i \rangle}{\langle \phi_i^* \phi_i \rangle} \tag{4}$$

The first variation of λ_i is

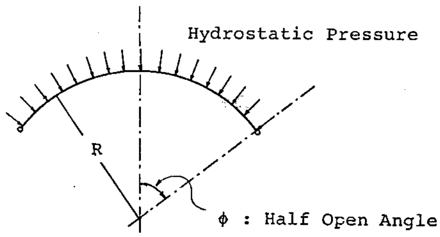
$$\delta\lambda_i = \frac{\langle \delta\phi_i^* (L\phi_i - \lambda_i \phi_i) + \delta\phi_i (L^* \phi_i^* - \lambda_i^* \phi_i^*) \rangle}{\langle \phi_i^* \phi_i \rangle} = 0 \tag{5}$$

which implies that the eigenvalues are stationary values.

The extremum property of Eq. (4) suggests the following approximate method of stability analysis. First, select two sets of trial functions $\phi_i (a_j)$ and $\phi_i^* (b_j)$ which satisfy the boundary conditions B and B*. Then, determine the unknown parameters a_j and b_j from equations of the type

$$\frac{\partial \lambda_i}{\partial a_j} = 0 \quad ; \quad \frac{\partial \lambda_i}{\partial b_j} = 0 \tag{6}$$

This procedure is analogous to the Rayleigh-Ritz method for conservative systems.



EI : Bending stiffness

A : Sectional area

i : = $\sqrt{I/A}$

Fig. 1. A Circular Arch

Numerical Analysis for a Circular Arch

Let us consider a circular arch shown in Fig. 1 and represent the governing equations and boundary condition as follows.

$$EAR^2\left(v''-w'+\frac{w' \cdot w''}{R}\right)+EI\,(w'''+v')=0 \tag{7}$$

$$\begin{aligned} &EAR^2\Big\{-v'+w+\frac{(w')^2}{2R}-\frac{1}{R}(w'v''+v'w''-ww'') \\ &-\frac{3}{2R^2}w''(w')^2\Big\}+EI\,(w^{iv}+v''')=qR^4 \end{aligned} \tag{8}$$

$$w=v=w''+v'=0 \quad at \quad \phi=\pm\phi_\circ \tag{9}$$

where $(\cdots)'=\frac{d}{d\phi}(\cdots)$. Since the critical point of the problem in question belongs to the divergence-type, the inertia terms in the above equations have been omitted. The hydrostatic pressure of follower type within an infinitesimal deflection may be represented by

$$q=q_0\left\{1-\triangle\frac{(w''+v')}{R}\right\} \tag{10}$$

Here $\triangle=0$ corresponds to the ordinary external pressure, remaining the initial load direction and $\triangle=1$ denotes the follower type.

The adjoint system (L*,B*) of this problem, which is obtained by substituting Eqs. (7), (8) and (9) into Eq. (1), has the property that L and L* are different but B is coincident with B*, i.e.,

$$w^*=v^*=w^{*''}+v^{*'}=0 \qquad at \ \phi=\pm\phi_0 \tag{11}$$

From this reason, we adopt the trial functions of the type

$$w=w^*=\sum^{N}a_n w_n, \quad v=v^*=\sum_{n=1}^{N'}b_n v_n \tag{12}$$

where

$$w_n = \{(2n+3)\phi_0^4-(4n+2)\phi_0^2\phi^2+(2n-1)\phi^4\}\ \phi^{n-1} \tag{13}$$

and

$$v_n = (\phi_0^4-2\phi_0^2\phi^2+\phi^4)\cdot\phi^{n-1} \tag{14}$$

Numerical analysis were carried out for $N=N'=8$ and the results are depicted in Fig. 2, which shows relation between shape parameter α and critical load. When q_{cr}^F and q_{cr}^0 correspond to $\triangle=1$ and $\triangle=0$, respectively, the ratio q_{cr}^F/q_{cr}^0 approximately equals to 0.52 in the almost region of shape parameter .

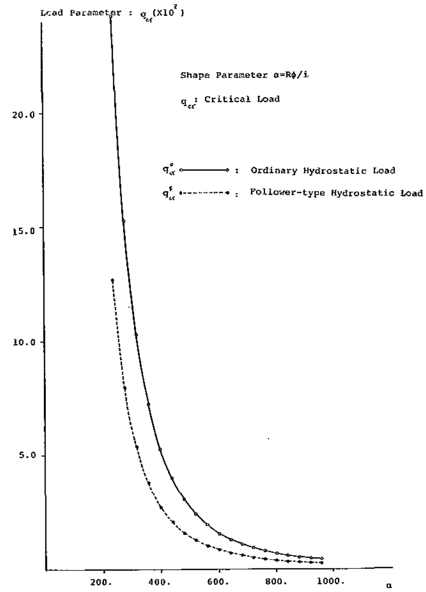


Fig. 2. Buckling Loads

Pursue of Pressure Deflection Curves

The previous analysis was approached from the standpoint of the linear eigen-value problem. In this section, pressure-deflection curves are pursued by the load incremental method in order to check the accuracy of the results shown in Fig. 2. The Galerkin method is applied to Eqs. (7) - (10), using the assumed deflection modes of Eq. (12). The results are shown in Figs. 3 and 4. These graphs shows that the pressure deflection curve keeps nearly linear in the pre-buckling region and the buckling pressures are coincident with the previous results. Furthermore, Fig. 3 shows the existence of the bifurcation phenomenon from $n=0$ to $n=3$ which is never occured in the case of the ordinary external pressure.

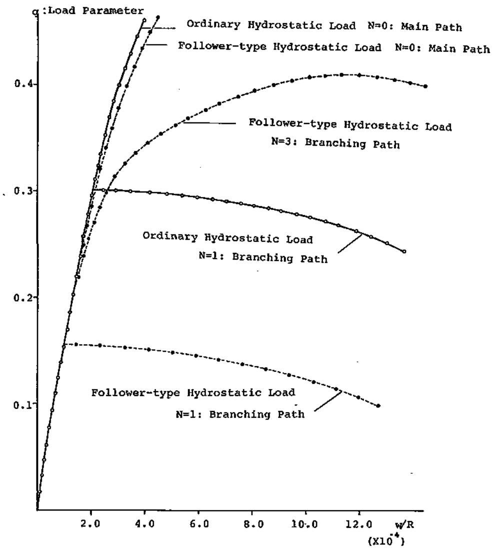


Fig. 3. Load Deflection Curve (α=200)

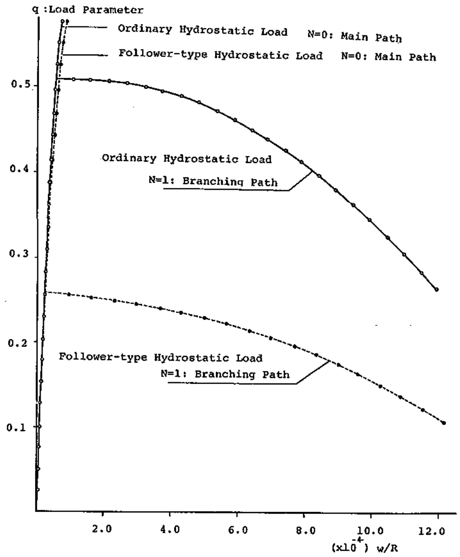


Fig. 4. Load Deflection Curve (α=400)

Effect of the Axial Strain ε_s on the Buckling Pressure

Let us obtain the buckling pressure in both the case of $\varepsilon_s=0$ and $\varepsilon_s\neq0$, by assuming $w=a\sin\pi\phi/\phi_0$, $w_0=a_0\cos\pi\phi/\phi_0$, $v=b\cos\pi\phi/\phi_0$, $v_0=b_0\sin\pi\phi/\phi_0$. Here w_0 and v_0 represent the pre-buckling deflections and w and v incremental deflections just after buckling. From the eigenvalue calculation,

$$q_{cr}=\frac{EI}{R^3}\left\{\left(\frac{\pi}{\phi_0}\right)^2-1\right\}\Big/\left[1+\triangle\cdot\frac{32}{15}\frac{a_0}{\pi R}\left\{\left(\frac{\pi}{2\phi_0}\right)^2-1\right\}\right] \quad for\ \varepsilon_s=0 \qquad (15)$$

and

$$q_{cr}=\frac{1}{2b_{11}b_{22}}\cdot\left\{-(a_{22}b_{11}+a_{11}b_{22}-a_{12}b_{21})\pm\sqrt{\left[(a_{22}b_{11}+a_{11}b_{22}-a_{12}b_{21})^2-4b_{11}b_{22}(a_{11}a_{22}-a_{12}a_{21})\right]}\right\} \quad for\ \varepsilon_s\neq0 \qquad (16)$$

where

$$a_{11}=-4/3\cdot\{1+(\pi/\phi_0)^2\cdot k\} \qquad k=I/R^2A$$

$$a_{12}=-(1+k)\cdot(\phi_0/4)\cdot(\pi/\phi_0)^2$$

$$a_{21}=-\phi_0\{1+k(\pi/\phi_0)^4\}$$

$$a_{22}=-1/3\cdot\{4+k(\pi/\phi_0)^2\}$$

$b_{11} = a_0 \cdot (\phi_0/8) \cdot (\pi/\phi_0)^3/R$

$b_{21} = a_0 \cdot (12/5) \cdot (\pi/\phi_0)/R + \triangle \cdot \phi_0(\pi/\phi_0)^2 R/EA$

$b_{22} = a_0 \cdot (\phi_0/8) \cdot (\pi/\phi_0)^3/R + \triangle \cdot (4/3)\mathrm{R}/EA$

Table 1 : Comparison of Buckling Load

	$\varepsilon_s = 0$	$\varepsilon_s \neq 0$
$\Delta = 0$	1.10×10^4	1.11×10^4
$\Delta = 1$	1.05×10^4	4.65×10^3

($\alpha = 200$)

Numerical results in the case of $\alpha = 200$ are shown in Table 1. In this case, q_{cr}^F / q_{cr}^0 is about 0.41 and the effect of the axial strain ε_s in the critical pressure is very large when we consider the follower type pressure load.

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