BRIEF NOTE

ENERGY CRITERIA FOR DYNAMIC BUCKLING OF SHALLOW STRUCTURES UNDER RECTANGULAR LOADING

by

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Introduction

Energy criteria have been often used for the establishment of the lower bound on the dynamic buckling load under step or impulse loading.[1,2] Along these lines, this paper presents an energy criterion for the dynamic buckling problems under the rectangular loading, based on the observation of the total potential energy surfaces for various load levels. Its energy criterion enables us to make a relation between the critical load level and time duration of the rectangular loading. In the theoretical formulation,an inequality of Liapunov's function is adopted to estimate the upper bound of the total energy, i.e. the sum of the potential and the kinetic energy, along the time history. The calculated results for the shallow shells under the rectangular loading will be presented.

Concepts of Energy Criteria

In order to clarify the concepts of the energy criteria, let us consider a nonlinear one-degree-of-freedom system under step or rectangular loading.

(a) Energy criterion to the step loading[1]

Fig. 1 shows a typical load-deflection curve and a family of the total potential energy curves for various load levels. Suppose a ball dropped from the origin of these curves, it is easy to understand that the deflection will be limited to a region in the vicinity of the origion as long as $q < q_D$. But, if q is increased above q_D, the ball will move away from the origin. Therefore, q_D is defined as the dynamic buckling load for the step loading.

(b) Energy criterion to the rectangular loading

For $t < 0$, a ball is supposed to be staying at the initial equilibrium point A on the total potential energy curve of $q = 0$

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as shown in Fig. 2. As soon as the load is applied, the ball starts to roll down on the total potential curve of $q = q_D$. At $t = t_d$, the ball, which possesses the kinetic energy obtained in the interval $[0, t_d]$, removes from B to C with the change of the energy curves by unloading. Then the ball climbs up toward the maximum point D on the energy curve of $q = 0$ by expending the kinetic energy.

If the ball cannot reach the point D, the ball returns the initial equilibrium position A. This means no occurrence of the dynamic buckling. From its observation, a sufficient condition for stability against dynamic buckling can now be derived as follows including the case of multi-degrees-of-freedom systems: If the total energy at $t = t_d$ does not exceed the minimum value of the relative maximum points or saddle points near the initial state on the strain energy curve of $q = 0$, the system is stable.[3]

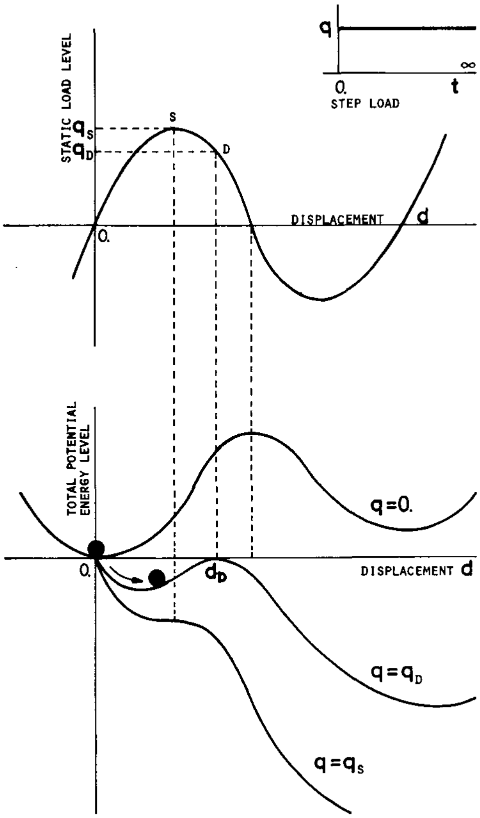

Fig. 1: Step Load and Energy Surfaces

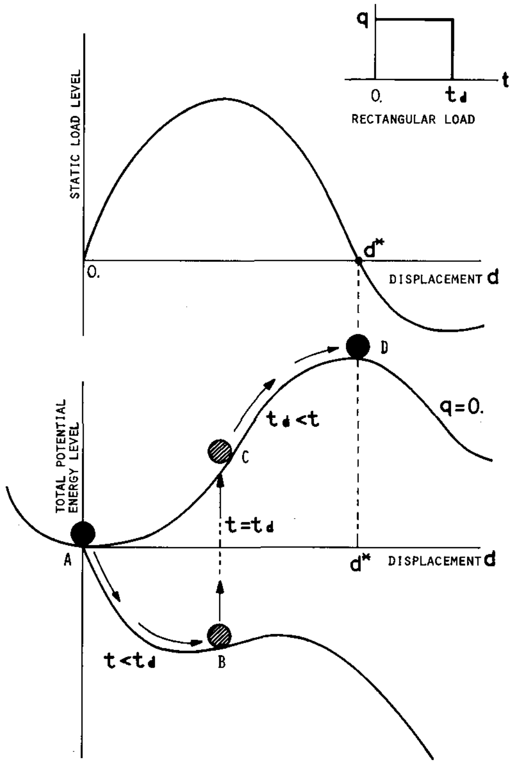

Fig. 2: Rectangular Load and Energy Surfaces

Analytical Formulation

Let us consider the nonlinear equations of motion:

$$\mathbf{M}\ddot{\mathbf{d}} + \mathbf{K}\mathbf{d} + \mathbf{g}(\mathbf{d}) = \mathbf{q} \tag{1}$$

where $\mathbf{M}$, $\mathbf{K}$, $\mathbf{d}$, $\mathbf{g}(\mathbf{d})$ and $\mathbf{q}$ are mass matrix, stiffness matrix, displacement vector, nonlinear terms and load vector, respectively.

If we introduce the function

$$V = \dot{\mathbf{d}}^T \mathbf{M} \dot{\mathbf{d}} + \mathbf{d}^T \mathbf{K}\mathbf{d} + 2U(\mathbf{d}) \quad , \quad \frac{\partial U}{\partial \mathbf{d}} = \mathbf{g}(\mathbf{d}) \tag{2}$$

which is propotional to the total energy, dV/dt ($\equiv \dot{V}$) along the solution of Eq. (1) takes the form:

$$\dot{V} = 2\dot{\mathbf{d}}^T \mathbf{q} \tag{3}$$

Then from the schwartz inequality

$$\begin{aligned} \dot{\mathbf{d}}^T \mathbf{q} &= \dot{\mathbf{d}}^T \mathbf{M}^{-1} \mathbf{M}\,\mathbf{q} \leqq \{\mathbf{q}^T \mathbf{M}^{-1} \mathbf{q}\}^{\frac{1}{2}} \{\dot{\mathbf{d}}^T \mathbf{M} \dot{\mathbf{d}}\}^{\frac{1}{2}} \\ &\leqq \mu(t) V^{\frac{1}{2}}(t) \end{aligned} \tag{4}$$

with $\mu(t) = \{\mathbf{q}^T \mathbf{M}^{-1} \mathbf{q}\}$, it follows that the function satisfies the differential inequality

$$\dot{V}(t) \leqq 2\mu(t) V^{\frac{1}{2}}(t) \, . \tag{5}$$

Integration of Eq. (5) yields

$$V^{\frac{1}{2}}(t) \leqq V^{\frac{1}{2}}(o) + \int_0^{t_d} \mu(t)\, dt \, . \tag{6}$$

Inequality (6) gives an upper bound on the function $V(t)$ along the solution of Eq. (1).

Hence, the sufficient condition for stability, which mentioned in the previous section, can now be written as

$$V^{\frac{1}{2}}(o) + \int_0^{t_d} \mu(t)\, dt \leqq [\mathbf{d}^{*T} \mathbf{K}\, \mathbf{d}^{*} + 2U(\mathbf{d}^{*})]^{\frac{1}{2}} \tag{7}$$

where $\mathbf{d}^*$ represents displacement vector which gives the minimum value of the relative maximum points or saddle points of the strain energy surfaces for q = 0 as mentioned in the sufficient condition for stability. In other words, $\mathbf{d}^*$ corresponds to the critical equilibrium point in the vicinity of the initial state (see Fig. 2), and can be obtained by using the static load-deflection curves.

In the case of the rectangular loading with the time duration of t_d, Eq. (7) becomes

$$\mu \leqq [\, \mathbf{d}^{*T}\mathbf{K}\,\mathbf{d}^* + 2U(\mathbf{d}^*)\,]^{\frac{1}{2}} / t_d \qquad (8)$$

which gives the relation between the critical load level and time duration since the critical load level $\mathbf{q}_{cr}$ is determined from $\mu_{cr} = \{\, \mathbf{q}^T \mathbf{M}^{-1} \mathbf{q}\,\}$.

Illustrative Examples

Consider the simply supported shallow shells with the constrained inplane displacements shown in Fig. 3.
The following nonlinear equation of motion and compatibility condition are adopted[4].

$$D\nabla^4 W - \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 W}{\partial x^2} + 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 W}{\partial x \partial y} - \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 W}{\partial y^2}$$

$$-\frac{1}{R}\left(\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial x^2}\right) + \rho h \frac{\partial^2 W}{\partial t^2} - P = 0 \qquad (9)$$

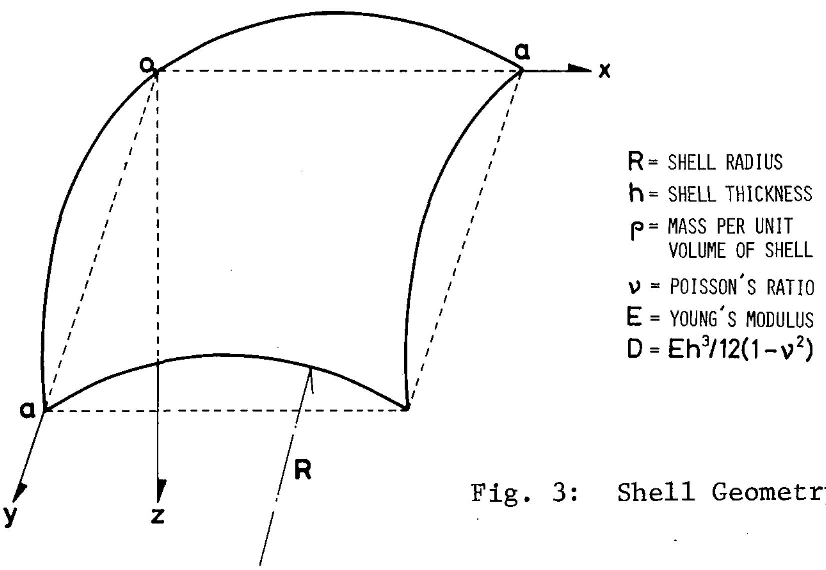

Fig. 3: Shell Geometry

$$\frac{1}{Eh}\nabla^4 F-\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2+\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}+\frac{1}{R}\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}\right)=0 \quad (10)$$

Let us represent the vertical displacement W and the load P as

$$W=W(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}, \quad P=P_0\sin\frac{\pi x}{a}\sin\frac{\pi y}{a} \quad (11)$$

and introduce the nondimensional quantities:

$$A=W/h, \quad \tau=\pi^2 h\sqrt{\frac{E}{\rho}}\,t\,/\,a^2\sqrt{3(1-\nu^2)}$$

$$\lambda=a^2/\pi^2 Rh, \quad q=3a^4(1-\nu^2)P_0/\pi^4 Eh^4\,. \quad (12)$$

If the Galerkin method is applied to Eq. (9) after obtaining the stress function F from Eq. (10), the equation of motion with one degree of freedom corresponding to Eq. (1) takes the form:

$$\frac{d^2 A}{d\tau^2}+(1+k_1\lambda^2)A-k_2\lambda A^2+k_3 A^3=q \quad (13)$$

where

$$k_1=6.06\,,\quad k_2=5.48 \text{ and } k_3=1.32\,.$$

The existence of the static load q_L for $t < 0$ was considered in the present numerical analysis to examine the influence of the dead load on the dynamic critical load levels. In its case, it was necessary for the reference coordinate system of Eq. (1) to be transformed to the load level q_L along the static equilibrium path before we used the analytical procedure mentioned in the previous section.

Fig. 4 shows the critical load levels under two kinds of step loadings ($t_d=\infty$) for various shape parameters (λ is defined in Eq. (12)). In the figures, q_s , q_L , q_D and q_D^* denote the static buckling load, the static load for $t < 0$, the dynamic step load in the case of $q_L = 0$ and the additional step load, respectively.

Fig. 5 and 6 show the relations between the dynamic critical load level ($q_L + q_D^*$) and the time duration parameter (T_d/T_0) for $\lambda = 5$ and 10 . T_0 represents the natural period of free vibration at the point of the static load level q_L. Present results are good agreement with the numerical integration results by means of the Runge-Kutta-Gill method as shown in figures.

A defect of the present approach is that curves by the energy

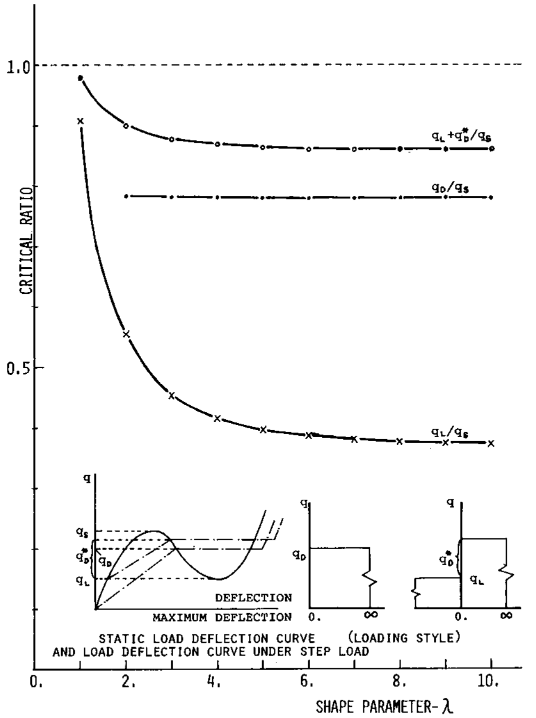

Fig. 4: Dynamic Buckling Load Levels for Step Loading

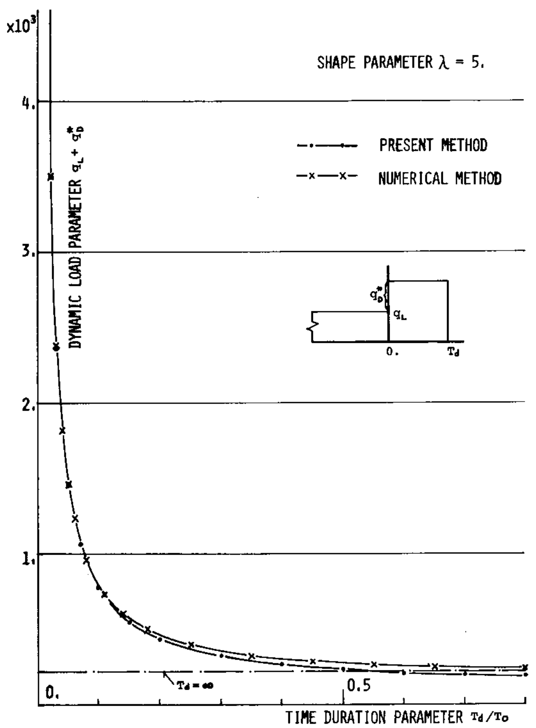

Fig. 5. Dynamic Buckling Load Levels for Rectangular Loading (λ=5)

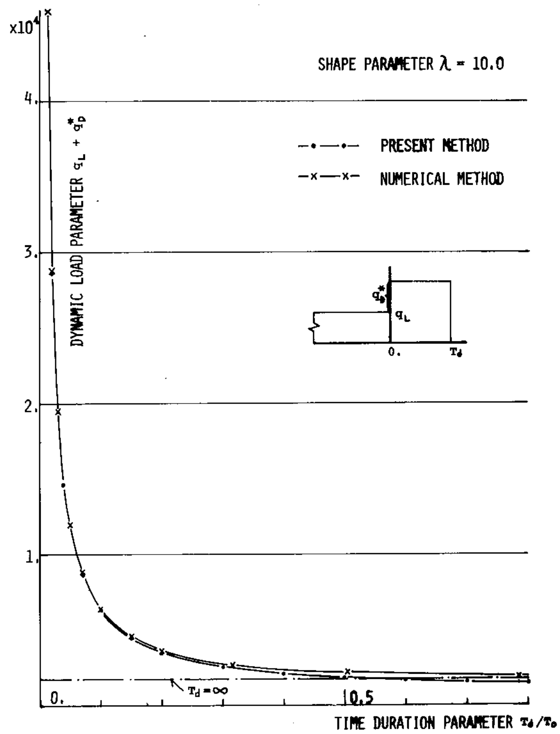

Fig. 6: Dynamic Buckling Load Levels for Rectangular Loading (λ=10)

criterion have a tendency to approach to zero as time duration increases, which makes a sever design criterion. So, in the region where T_d is comparatively large, we had better to adopt the critical load levels by the step loading, which are plotted by dot-dashed lines in these figures.

Numerical examples of multi-degrees-of-freedom systems have been presented in [5].

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