FINITE ELEMENT ANALYSIS OF THE TSUNAMI PROBLEM

by

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Summary

As far as the authors know, concrete theoretical basis of the tsunami problem has not yet been established because of the complex nonlinear behavior although much attempt has been made in the past. Today, however, it is known that the nonlinear dispersive wave equations can describe such a wave as tsunami nicely. The KdV equation might be one of typical examples of these equation and it is derived under the assumption that the weak nonlinearity and the weak dispersion are well balanced. In this paper, using such non-linear dispersive equations including KdV equation, finite element analysis of the tsunami problem is proposed and results of numerical analysis on some simple problems are shown in good agreement with the results of previous studies.

Introduction

It is known that the tsunami will take place due to sudden disclocation of the ocean bottom caused by the earthquake. It is believed from witness' reports that the tsunami consists of three independent waves in almost all the cases, and in some case the first wave may be of the highest, while in other case the second wave may be of the highest, and its propagation in the far-field could be explained by the solitary wave theory.

For the last 10 years keen attention has been focused on the analysis of the KdV equation in many fields of physical science and engineering. The KdV equation was derived by Korteweg and de Vries in the courses of the theoretical study on the shallow water wave problem in 1895, and they were successful in obtaining the cnoidal wave solution. This is a steady solution of the KdV equation and the solitary wave solution can be derived by making the period of the cn-wave solution infinitely large. The original form of the KdV equation which describes the small but finite amplitude wave is given as follows:

$$\frac{\partial \eta}{\partial t} + C_0 \left(\frac{\partial \eta}{\partial x} + \frac{3}{2h} \eta \frac{\partial \eta}{\partial x} + \frac{h^2}{6} \frac{\partial^3 \eta}{\partial x^3} \right) = 0 \qquad (1)$$

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where η is the wave height, h is the uniform depth, g is the gravitational acceleration, and $C_0 = \sqrt{gh}$.

In 1965, using the KdV equation in the plasma physics, N.J. Zabusky and M.D. Kruskal showed by their numerical experiment that these waves are so stable although they interact and observing such a characteristics like particles they called a solitary wave 'SOLITON'.

In 1966 D.H. Peregrine first proposed the following equation to analyse the 'bore' problem numerically:

$$\frac{\partial u}{\partial t} + \left(C_0 + \frac{3}{2} u \right) \frac{\partial u}{\partial x} - \frac{h^2}{6} \frac{\partial^3 u}{\partial t \, \partial x^2} = 0$$

$$\eta = \frac{h}{C_0} \left(u + \frac{1}{4} \frac{u^2}{C_0} - h^2 \frac{\partial^2 u}{\partial x^2} \right) \tag{2}$$

where u is the mean horizontal velocity. Eq. (2) is of the same order of approximation as the KdV equation.

In 1973 J.L. Hammack conducted the numerical analysis of the tsunami propagation in the far-field by using above equations and he confirmed good agreement between the results of numerical calculation and experiments. In other words he found that when the rise up of the water surface takes place in the shallow water of the constant depth in the channel due to the sudden vertical deformations or some another reasons, wave propagation will occur and they will resolve into several solitons and tails as it propagates.

The nonlinear theory which include no dispersive effect as well as the linear theory which is related to the Cauchy-Poisson wave theory has been proposed in the past to predict the tsunami propagation, but none of them was successful to explain the phenomenon that the several solitons will appear when the sea bed deforms vertically in an instant.

However, eqs. (1) and (2) which can be derived by assuming that the weak nonlinearity and the weak dispersion are appropriately balanced can account for the phenomenon discussed above.

In the following sections, derivations of eqs. (1), (2) and other equations will be briefly described and the numerical results will be also shown with some discussions.

For numerical analysis of this kind of problems, the finite difference method has been mainly adopted, but the present authors believe that the finite element method could be more practical and flexible to compare with the finite difference method in analysis of complicated boundary value problems or extended application of the proposed theory to the two dimensional problems.

1. Derivations of the KdV and other analogous equations

If it is assumed that the fluid is incompressible and the flow is irrotational, a velocity potential $\phi = \phi\ (x, y, t)$ can be introduced, hence from the continuity equation the field equation is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \text{in } V \tag{3}$$

The boundary conditions on the free surface are

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} \tag{4}$$

and

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left\{ \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 \right\} + g\,\eta = 0 \tag{5}$$

$$\text{at } y = h + \eta(x, t)$$

where it is assumed that the surface tension can be neglected. The boundary condition at the bottom can be assumed as follows;

$$\frac{\partial \phi}{\partial y} = 0 \qquad \text{at } y = 0 \tag{6}$$

in case where the wave behaviour in the downstream region is considered. Eq. (6) may not be applicable to the region of the wave origin, instead the equation including the time-dependent bottom deformation should be used.

Using the characteristic quantities h and C_0, eqs. (3) ∿ (6) are non-dimensionalized as follows:

$$\frac{\partial^2 \phi}{\partial y} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \text{in } V \tag{7}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x}\frac{\partial \phi}{\partial x} \qquad \text{at } y = 1 + \eta \tag{8}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left\{ \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 \right\} + \eta = 0 \qquad \text{at } y = 1 + \eta \tag{9}$$

$$\frac{\partial \phi}{\partial y}=0 \qquad \text{at} \quad y=0 \tag{10}$$

It should be mentioned that the same notations occasionally will be used both in dimensional and non-dimensional forms because the resulting confusion may seldom occur.

The nonlinear dispersive waves in the shallow water is considered. Introducing the very small parameter ε which represents the weak nonlinearity and the weak dispersion of the medium, the following coordinates transformation is applied to eqs. (7) ∿ (10).

$$\xi=\varepsilon^{1/2} x, \quad \tau=\varepsilon^{1/2} t, \quad y=y \tag{11}$$

Then the field equation (7) and the boundary conditions (8), (9), (10), take, the following forms respectively:

$$\varepsilon \frac{\partial^2 \phi}{\partial \xi^2}+\frac{\partial^2 \phi}{\partial y^2}=0 \tag{12}$$

$$\frac{\partial \phi}{\partial y}=\varepsilon^{1/2} \frac{\partial \phi}{\partial \tau}+\varepsilon \frac{\partial \eta}{\partial \xi} \frac{\partial \phi}{\partial \xi} \qquad \text{at} \quad y=1+\eta \tag{13}$$

$$\varepsilon^{1/2} \frac{\partial \phi}{\partial \tau}+\frac{1}{2}\left\{\varepsilon\left(\frac{\partial \phi}{\partial \xi}\right)^2+\left(\frac{\partial \phi}{\partial y}\right)^2\right\}+\eta=0 \qquad \text{at} \quad y=1+\eta \tag{14}$$

$$\frac{\partial \phi}{\partial y}=0 \qquad \text{at} \quad y=0 \tag{15}$$

Next, it is assumed that the dependent variables η and ϕ are expanded into the power series of ε as follows:

$$\eta=\varepsilon \eta^{(1)}(\xi, \tau)+\varepsilon^2 \eta^{(2)}(\xi, \tau)+\cdots\cdots \tag{16}$$

$$\phi=\varepsilon^{1/2}\left\{\phi^{(1)}(\xi, y, \tau)+\varepsilon \phi^{(2)}(\xi, y, \tau)+\cdots\cdots\right\} \tag{17}$$

Substituting eq. (17) into eqs. (12) and (15), and arranging them in the power of ε , the integrated terms of the velocity potential with respect to y are finally given as follows:

$$\phi^{(1)}=\phi^{(1)}(\xi,\ \tau) \tag{18-a}$$

$$\phi^{(2)}=-\frac{y^2}{2}\frac{\partial^2\phi^{(1)}}{\partial\xi^2}+F(\xi,\ \tau) \tag{18-b}$$

$$\frac{\partial\phi^{(3)}}{\partial y}=\frac{y^3}{6}\frac{\partial^4\phi^{(1)}}{\partial\xi^4}-y\frac{\partial^2 F}{\partial\xi^2} \tag{18-c}$$

....

where F (ξ , τ) is the arbitrary function. It should be mentioned that eq. (18-a) shows that $\phi^{(1)}$ to the lowest order of ε is independent of y, that is, corresponds to the basic assumptions in the linear long wave theory that the vertical component of the velocity is zero and the pressure distribution is equal to the hydrostatical one.
Substituting eqs. (16) and (17) into the surface boundary conditions of eqs. (13) and (14), and using the relations of eqs. (18-a, b,c), the following equations are obtained:

$$-\frac{\partial^2\phi^{(1)}}{\partial\xi^2}=\frac{\partial\eta^{(1)}}{\partial\tau} \tag{19-a}$$

$$-\eta^{(1)}\frac{\partial^2\phi^{(1)}}{\partial\xi^2}+\frac{1}{6}\frac{\partial^4\phi^{(1)}}{\partial\xi^4}-\frac{\partial^2 F}{\partial\xi^2}=\frac{\partial\eta^{(2)}}{\partial\tau}+\frac{\partial\eta^{(1)}}{\partial\xi}\frac{\partial\phi^{(1)}}{\partial\xi} \tag{19-b}$$

and

$$\frac{\partial\phi^{(1)}}{\partial\tau}+\eta^{(1)}=0 \tag{20-a}$$

$$-\frac{1}{2}\frac{\partial^2\phi^{(1)}}{\partial\xi^2\,\partial\tau}+\frac{\partial F}{\partial\tau}-\frac{1}{2}\left(\frac{\partial\phi^{(1)}}{\partial\xi}\right)^2+\eta^{(2)}=0 \tag{20-a}$$

....

Noting eqs. (19-a) and (20-a), and defining the horizontal velocity by $u^{(1)}=\partial\phi^{(1)}/\partial\xi$, the long wave equations in the linear theory are derived as follows:

$$\frac{\partial \eta^{(1)}}{\partial \tau} + \frac{\partial u^{(1)}}{\partial \xi} = 0$$

$$\frac{\partial u^{(1)}}{\partial \tau} + \frac{\partial \eta^{(1)}}{\partial \xi} = 0 \tag{21}$$

Now the elevation from the statical water level and the mean velocity potential are defined approximately, and they are given by the following equations respectively;

$$\bar{\eta} = \varepsilon \eta^{(1)} + \varepsilon^2 \eta^{(2)}$$

$$\bar{\phi} = \frac{1}{1+\eta} \int_0^{1+\eta} \varepsilon^{1/2} (\phi^{(1)} + \varepsilon \phi^{(2)}) \, dy \tag{22}$$

$$= \varepsilon^{1/2} \{ \phi^{(1)} + \varepsilon (-\frac{1}{6} \frac{\partial^2 \phi^{(1)}}{\partial \xi^2} + F) \}$$

From eqs. (19-a,b), using the above variables, the following equation is obtained by neglecting the higher order terms of ε whose power is greater than 2.5

$$\varepsilon^{1/2} \frac{\partial \bar{\eta}}{\partial \tau} + \varepsilon \frac{\partial^2 \bar{\phi}}{\partial \xi^2} + \varepsilon \frac{\partial}{\partial \xi} (\bar{\eta} \frac{\partial \bar{\phi}}{\partial \xi}) = 0 \tag{23}$$

This is the continuity equation.
From eqs. (20-a,b), the momentum equation is expressed as follows:

$$\bar{\eta} + \varepsilon^{1/2} \frac{\partial \bar{\phi}}{\partial t} + \frac{\varepsilon}{2} (\frac{\partial \bar{\phi}}{\partial \xi})^2 - \frac{\varepsilon^{3/2}}{3} \frac{\partial^3 \bar{\phi}}{\partial \xi^2 \partial \tau} = 0 \tag{24}$$

With changing back to x and t, and introducing the mean horizontal velocity defined by $u = \partial \bar{\phi} / \partial x$, the basic equations in the shallow water problem are obtained as follows:

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} [(1+\eta) u] = 0 \tag{25}$$

$$\frac{\partial u}{\partial t} + \frac{\partial \eta}{\partial x} + u \frac{\partial u}{\partial x} = \frac{1}{3} \frac{\partial^3 u}{\partial t \partial x^2} \tag{26}$$

where the bar of η is omitted.
In the above two equations when a wave propagation in one-direction is only considered, the following approximate relations between u and η can be obtained:

$$\eta \doteqdot u + \frac{1}{4} u^2 + \frac{1}{6} \frac{\partial^2 u}{\partial t \partial x} \tag{27-a}$$

$$u \doteqdot \eta - \frac{1}{4} \eta^2 - \frac{1}{6} \frac{\partial^2 \eta}{\partial t \partial x} \tag{27-b}$$

Substituting these relations into eqs. (25) and (26), and neglecting higher order terms, the following equations are derived:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \frac{3}{2} u \frac{\partial u}{\partial x} = \frac{1}{6} \frac{\partial^3 u}{\partial t \partial x} \tag{28-a}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} + \frac{3}{2} \eta \frac{\partial \eta}{\partial x} = \frac{1}{6} \frac{\partial^3 \eta}{\partial t \partial x^2} \tag{28-b}$$

Assuming that from the linear equations between u and η shown by eq. (21), the following approximate relation of the differential operators can be adopted:

$$\frac{\partial}{\partial t} \doteqdot - \frac{\partial}{\partial x} \tag{29}$$

eq. (27-a) can be expressed by

$$\eta = u + \frac{1}{4} u^2 - \frac{1}{6} \frac{\partial^2 u}{\partial x^2} \tag{30}$$

J.L. Hammack (1973) obtained the good numerical results by using eqs. (28-a) and (30).
Replacing $\partial / \partial t$ of the left-handside of eq. (28-b) by $- \partial / \partial x$, the well-known KdV equation can be derived as follows:

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} + \frac{3}{2} \eta \frac{\partial \eta}{\partial x} + \frac{1}{6} \frac{\partial^3 \eta}{\partial x^3} = 0 \tag{31}$$

From eqs. (25) and (26), the Boussinesq equation can be also derived.

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} = \frac{\partial^2}{\partial x^2} \left(\frac{3}{2} \eta^2 + \frac{1}{3} \frac{\partial^2 \eta}{\partial x^2} \right) \tag{32}$$

2. Theoretical formulation by means of the method of weighted residuals

For analysis of the propagation problem of the tsunami waves in the downstream region, the present authors adopt the KdV equation, because eq. (28-a) and eq. (31) are of the same order of approximation and both equations have solitons as their stationary solution.

First of all, the following transformation is applied to eq. (31);

$$r = x - t$$

$$\tau = t/6 \qquad (33)$$

$$\zeta = \frac{3}{2}\eta(r,\ \tau)$$

With this transformation, the following generalized KdV equation can be obtained.

$$\frac{\partial \zeta}{\partial \tau} + 6\zeta\frac{\partial \zeta}{\partial r} + \frac{\partial^3 \zeta}{\partial r^3} = 0 \qquad (34)$$

The associated boundary conditions and initial condition can be given by the following equations respectively;

$$\lim_{|r|\to\infty} \zeta = \lim_{|r|\to\infty} \frac{\partial\zeta}{\partial r} = \lim_{|r|\to\infty} \frac{\partial^2\zeta}{\partial r^2} = 0 \qquad (35)$$

$$\zeta(r,0) = g(r) \qquad (36)$$

The solution for general initial-boundary value problems can be considered and in case where the following initial condition is selected:

$$g(r) = A\ \mathrm{sec}\,h^2\left(\sqrt{\frac{A}{2}}\,r\right)$$

it is not difficult to show that the solution of eq. (34) can be given by the following equation;

$$\zeta(r,\ \tau) = A\ \mathrm{sec}\,h^2\left[\sqrt{\frac{A}{2}}\,(r - 2A\tau)\right] \qquad (37)$$

where A is the amplitude of a given soliton and it can be seen that the phase velocity is proportional to the amplitude A.

In order to develop the finite element method of analysis of eq.

(34), eq. (34) should be modified to Galerkin type variational equation by using the method of weighted residuals, and such procedure will be briefly described as follows;

The following weighted residual integral of eq. (34) is considered:

$$\int_{-\infty}^{\infty} \delta\zeta \left(\frac{\partial \zeta}{\partial \tau} + 6\zeta\frac{\partial \zeta}{\partial r} + \frac{\partial^3 \zeta}{\partial r^3}\right) dr = 0 \tag{38}$$

in which the weight function $\delta\zeta$ should satisfy the prescribed boundary conditions. Preforming the integration by parts and applying the boundary conditions eq. (35), eq. (38) can be finally transformed into the following Galerkin equation:

$$\int_{-\infty}^{\infty} \left(\frac{\partial \zeta}{\partial \tau}\delta\zeta + \frac{\partial \zeta}{\partial r}\frac{\partial^2 \delta\zeta}{\partial r^2} - 3\zeta^2\frac{\partial \delta\zeta}{\partial r}\right) dr = 0 \tag{39}$$

Basing on this equation, finite element discretization now can be made in the following way. Since it is known that the soliton tends to become zero at a sufficient distance from its center and therefore it may be always possible to truncate the region of integration such that the error resulting from truncation can be neglected. The truncated region of integration is now devided into a finite number of elements, the displacement function of which is defined as follows:

$$\zeta = \lfloor \mathbf{N} \rfloor \{\zeta\} \tag{40}$$

where $\lfloor \zeta \rfloor$ is nodal displacement vector of a given element

$$\lfloor \zeta \rfloor = \lfloor \zeta_1 ,\ (\zeta,_{r})_1,\ (\zeta,_{rr})_1,\ \zeta_2, (\zeta,_{r})_2,\ (\zeta,_{rr})_2 \rfloor \tag{41}$$

and $\lfloor \mathbf{N} \rfloor$ is the shape function defined by the following equation: (See. Fig. 1)

$$\{\mathbf{N}\} = \left\{\begin{array}{l} (1-\xi)^3\ (3\xi^2+9\xi+8)/16 \\ (1+\xi)(1-\xi)^3\ (3\xi+5)\ell/32 \\ (1+\xi)^2\ (1-\xi)^3\ \ell^2/64 \\ (1+\xi)^3\ (3\xi^2-9\xi+8)/16 \\ (1+\xi)^3\ (1-\xi)(3\xi-5)\ell/32 \\ (1+\xi)^3\ (1-\xi)^2\ \ell^2/64 \end{array}\right\}$$

and

$$r = \lfloor (1-\xi)/2, (1+\xi)/2 \rfloor \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix}$$

$$\ell = r_2 - r_1 \quad (>0)$$

(42)

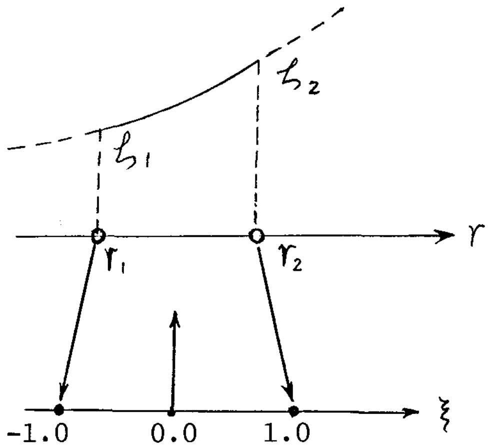

Fig. 1 Transformation of element shape function

Using the displacement function eq. (42) and following a standard technique of the finite element method, the matrix equation can be derived as follows:

$$[\widetilde{\mathbf{M}}_0]\{\dot{\zeta}\} + [\widetilde{\mathbf{M}}_2]\{\zeta\} - 3\int [\mathbf{M}_1]\{\zeta\}\lfloor \zeta \rfloor\{\mathbf{N}\}\,dr = 0 \qquad (43)$$

where

$$\left.\begin{aligned} &\{\dot{\zeta}\} = \frac{d}{d\tau}\{\zeta\} \\ &[\widetilde{\mathbf{M}}_0] = \int \{\mathbf{N}\}\lfloor \mathbf{N} \rfloor\,dr\,, \quad \lfloor \widetilde{\mathbf{M}}_2 \rfloor = \int \{\mathbf{N},_{rr}\}\lfloor \mathbf{N},_r \rfloor\,dr \\ &\text{and} \\ &[\mathbf{M}_1] = \{\mathbf{N},_r\}\lfloor \mathbf{N} \rfloor \end{aligned}\right\} \qquad (44)$$

For the time integration, the following scheme is adopted:

$$\{\zeta_n\} = \{\zeta_0\} \qquad \text{at } \tau = \tau_0$$

$$\{\zeta_{n+1}\} = \{\zeta_n\} + \{\Delta\zeta\} = \{\zeta_0\} + \{\Delta\zeta\} \qquad \text{at } \tau = \tau_0 + \Delta\tau \qquad (45)$$

$\lfloor \Delta\zeta \rfloor$ is the increment of $\lfloor \zeta_n \rfloor$ during the time increment $\Delta\tau$ from $\tau = \tau_0$ and the following scheme for time derivative is also adopted:

$$\{ \dot{\zeta} \} = \frac{2}{\Delta\tau} \{ \Delta\zeta \} - \{ \zeta_0 \} \tag{46}$$

This is an implicit and step by step time integration scheme, and the following matrix equation of incremental form can be derived by substituting eqs. (45), (46) into eq. (43) and neglecting the higher order term of $\lfloor \Delta\zeta \rfloor$:

$$\left(\frac{2}{\Delta\tau} [\widetilde{\mathbf{M}}_0] + [\widetilde{\mathbf{M}}_2] - 6 \int [\mathbf{M}_1] \{ \zeta_0 \} \lfloor \mathbf{N} \rfloor \, dr \right) \{ \Delta \mathbf{u} \}$$
$$= [\widetilde{\mathbf{M}}_0] \{ \dot{\zeta}_0 \} - [\widetilde{\mathbf{M}}_2] + 3 \int [\widetilde{\mathbf{M}}_1] \{ \zeta_0 \} \lfloor \zeta_0 \rfloor \{ \mathbf{N} \} \, dr \tag{47}$$

Solving this equation successively at every time step of $\Delta\tau$, the unsteady problem of the solitons can be studied.
It should be mentioned that in the present analysis an elementary scheme is adopted for time integration, and more elaborate scheme such as Newton-Raphson method should be employed in case where more accurate analysis is required.

Numerical examples and discussions

In order to show the validity of Galerkin equation eq. (38) so far derived and accuracy of the method of solution, three simple numerical examples will be given. Fig. 2 shows the result of numerical calculation which was made by using Newton-Raphson method in order to check validity of the proposed finite element method. It should be noted here that the analysis was made with respect to the coordinate fixed to a given soliton and result of this numerical analysis showed in good agreement with the exact solution. Fig. 3 shows a result of numerical analysis of a non-stationary problem by using the linearized expression for the displacement increment $\lfloor \Delta\zeta \rfloor$ to check parallel movement of a soliton and again accuracy of the analysis was duly shown by this simply example. As the third example, comparision was made on the result of the finite element analysis of two solitons' interaction with the analytical solution of the same problems. In the finite element analysis of this problem two solitons at t=0 are given by the equations in Fig. 4.

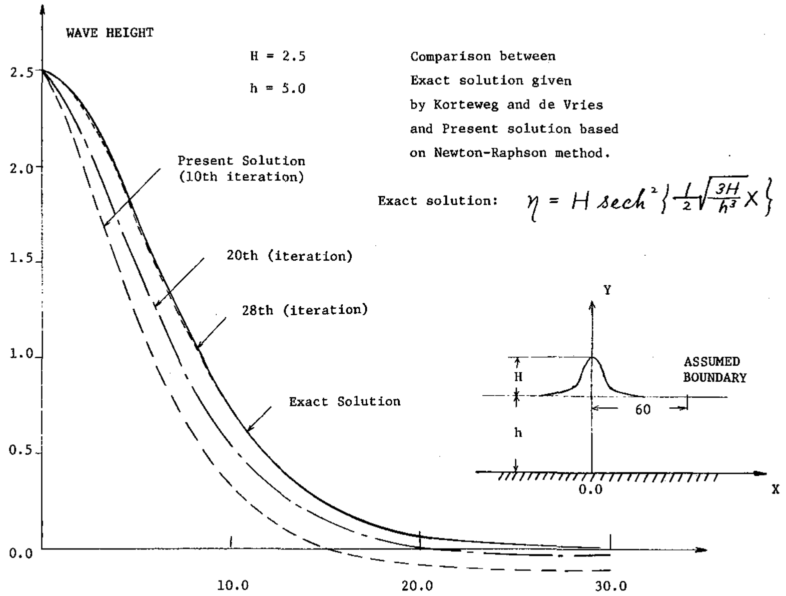

Fig. 2 Finite Element Solution of KdV Equation

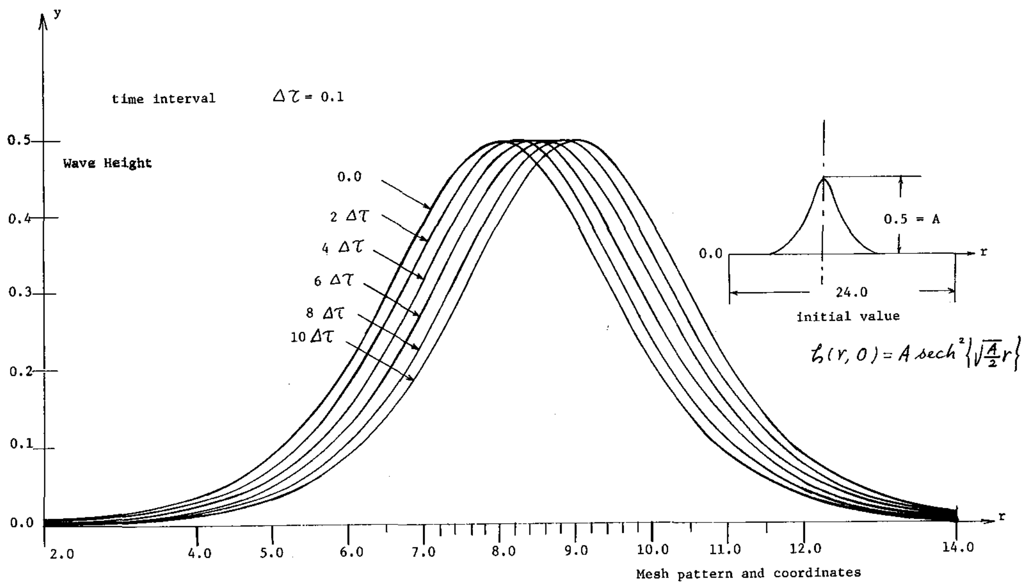

Fig. 3 A Simple Nonstationary Problem Analysed by Incremental Approach

It is also assumed that initially the soliton of higher amplitude was located behind the smaller one. With time going by, the former will soon overtake and pass the latter. They may influence each other in a complicated manner, but after interaction they will be separated each other again changing the order, but without changing their shape as well as amplitude. It should be mentioned here that phases of two solitons, however, may be changed between before and after their interaction. That is, the phase of the soliton of higher amplitude may increase while that of smaller amplitude may decrease.

In the numerical analysis, however, it is not easy to trace such variations. The result of the finite element analysis is shown in Fig. 4, while that of the analytical solution is shown in Fig. 5.

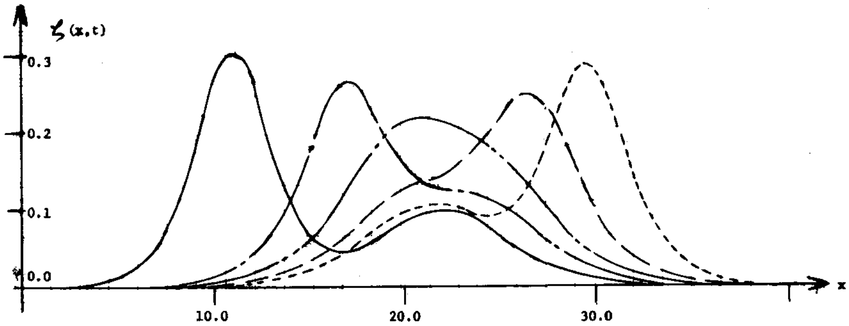

Fig. 4. Interaction of two solitons (finite element solution) (initial conditions;

$$\zeta(x,0) = \sum_{i=1}^{2} A_i \,\mathrm{sech}^2\left[\sqrt{\frac{A_i}{2}}(x-\theta_i)\right],$$ where $A_1=0.3$, $\theta_1=11.0$,

$A_2=0.1$ and $\theta_2=22.0$, linear superposition), time increment; $\Delta t=1.0$, and ——— at t=0.0, —·— at t=10 Δt, —··— at t=15Δt, — — — at t=20Δt, and ------ at t=25 Δt,

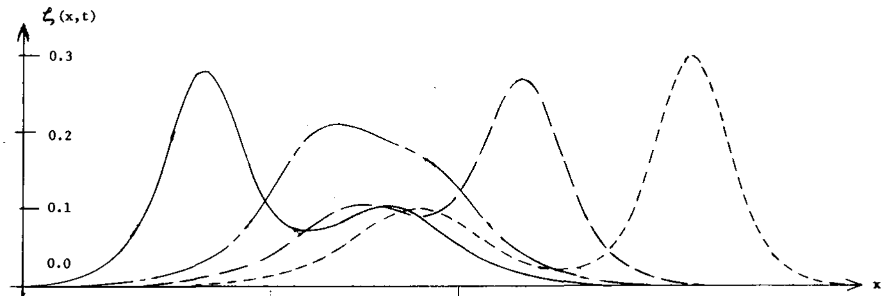

Fig. 5. Analytical solution of two solitons interaction by the inverse method [Segur (1973)]. The larger soliton's amplitude is 0.3 and the smaller is 0.1, and they are sufficiently separated at the initial state. T the basic time and Δt the time increment; Δt = 1.0, ——— at t = T, —·— at t = T + 11 Δt, — — — at t = T + 22 Δt, and ---------- at t = T + 37 Δt.

The analytical solution was obtained under the assumption that two solitons were sufficiently separated each other.
In order to cut the computing time, initial condition was given at the time as long as the principle of superposition of two solitons can be assumed although they are interacted, and because the initial conditions in these two solutions are different, strictly speaking it is not possible to compare the numerical result with the analytical solution, however, it can be observed that the nonlinear mutual interaction is nearly consistent each other.

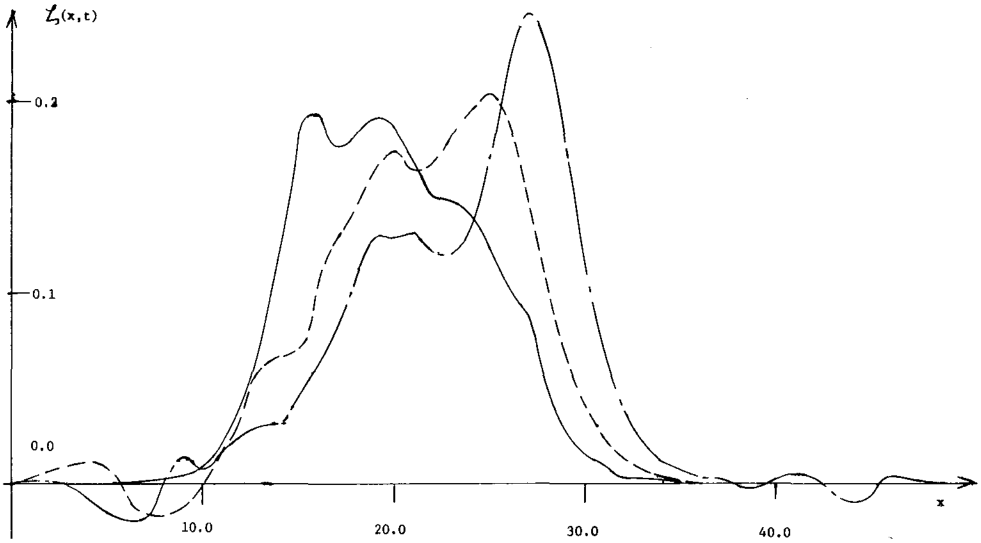

Fig. 6. Generation of solitons from an arbitrary initial state, initial values of $\zeta(x,0)$ at nodal points are given artificially, time increment; $\Delta t = 0.8$, ———— at $t = 0.0$, --------- at $t = 6\Delta t$, and ——-—— at $t = 12\Delta t$.

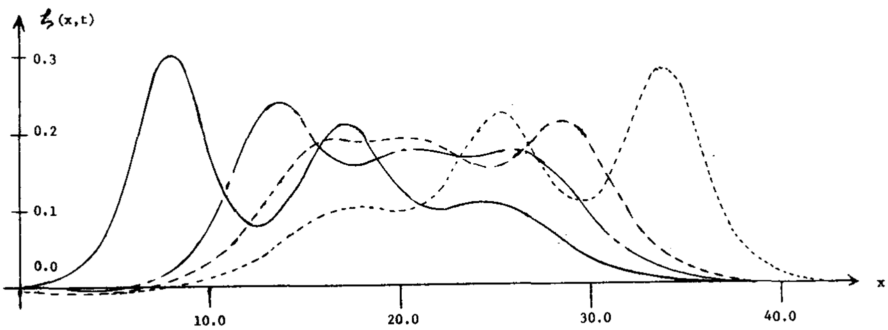

Fig. 7. Interaction of three solitons (initial) conditions; $\zeta(x,0) = \sum_{i=1}^{3} A_i \sec h^2 \left[\sqrt{\frac{A_i}{2}}(x-\theta_i)\right]$, where $A_1 = 0.3$, $\theta_1 = 11.0$, $A_2 = 0.2$, $\theta_2 = 20.0$, $A_3 = 0.1$, $\theta_3 = 28.0$), time interval; $t = 1.0$, ———— at $t = 0.0$, ——-—— at $t = 10\Delta t$,---- at $t = 15\Delta t$, and -------------- at $t = 25\Delta t$.

Basing on the study on the validity of the proposed method and accuracy of analysis, analyses of several examples simulating interaction of several solitons are made.
The generation of the solitons from the arbitrary initial condition is shown in Fig. 6. Behind the solitons, a 'tail' is created. In the present analysis Newton's forward difference scheme was adopted for calculation of initial values. This mesh division may not be accurate in the case of such tail analysis and the more elaborate scheme should be selected. Fig. 7 shows the interactions of three solitons. It can be observed that after their interaction they are separated again just like in case of two solitons' interaction shown by Fig. 4. It should be also mentioned that the 'bore' is created at the time of the interactions. This situation may suggest the new concept of 'tsunami' in explaining its change of wave form in shallow water. Fig. 8 shows the simulation when the bore is adopted as the initial condition. This bore is equivalent to the shock wave physically. From this figure, it can be seen that the front of the bore heaps up. For analysis of this case, the Galerkin equation which is different from eq. (39) was used because the first equation of the boundary conditions, eq. (35) is not same.

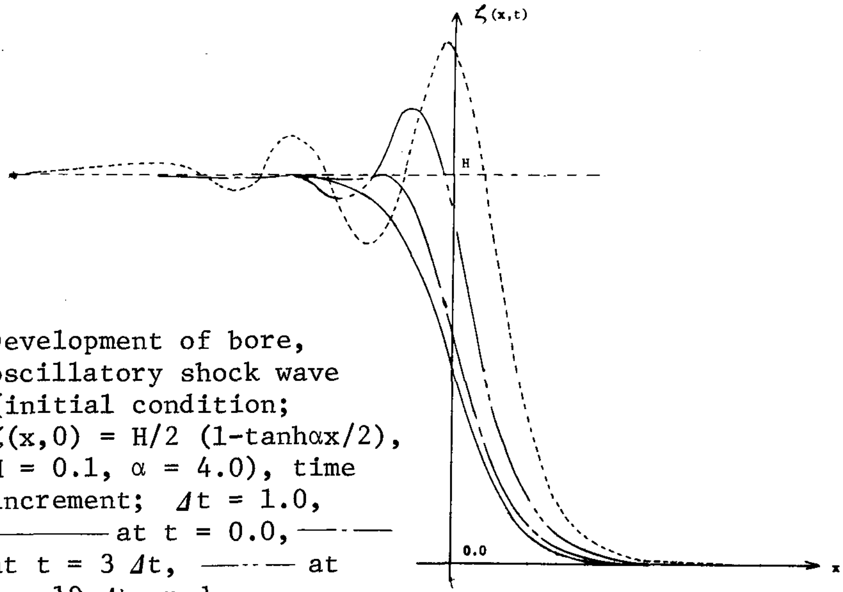

Fig. 8. Development of bore, oscillatory shock wave (initial condition; $\zeta(x,0) = H/2\ (1-\tanh\alpha x/2)$, $H = 0.1$, $\alpha = 4.0$), time increment; $\Delta t = 1.0$, ——— at $t = 0.0$, —·— at $t = 3\ \Delta t$, —··— at $t = 10\ \Delta t$, and -------- at $t = 18\ \Delta t$.

Using eqs. (25) and (26) studies are made on the interaction of two solitons of the same amplitude travelling in opposite direction and/or reflection of a soliton after its collision with a vertical wall. By introducing the weight functions $\delta\eta$ and δu and multiplying them to eq. (26) and eq. (25) respectively, Galerkin type variational equations are derived as described in the previous section. For the finite element analysis, the incremental approach is adopted for analysis of nonstationary problem and as the nodal parameters up to the first derivative of u and η are considered and Hermitian third order polynomials are used for the shape function of a given element. As for the initial value of the wave height, a soliton solution which is the solution of KdV equation, while the initial value of the velocity is calculated from eq. (27-b).

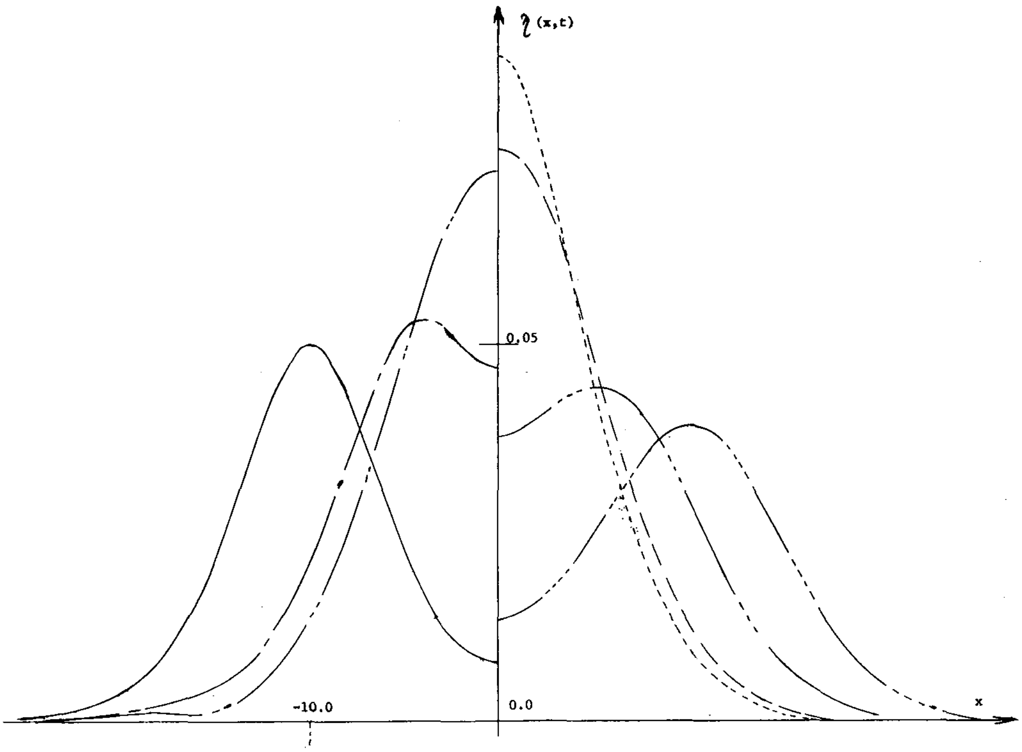

Fig. 9. Interaction of two solitons travelling in opposite directions (initial) conditions;

$$\eta(x,0)=\sum_{i=1}^{2} \mathrm{Hi}\ \mathrm{sech}^2\left[\sqrt{\frac{3H_i}{2}}(x-\theta_i)\right] \quad \text{where, } H_1=H_2=0.05,$$

$\theta_1=-10.0$, $\theta_2=10.0$), time inerement; t = 0.4878, ———— at t = 0.0, —— - —— at t = 10 Δt, —— -- —— at t = 14 Δt, ------------ at t = 19 Δt, ——————at t = 24 Δt, ——---—— at t = 32 Δt, and ——----—— t = 40 Δt.

R.E. Meyer (1963) showd by using Boussinesq equation eq. (32) that the principle of superposition is almost justified when the wave amplitude is small. And J.G.B. Byatt-Smith (1971) discussed that eq. (32) is correct as far as the first approximation is concerned and by deriving an equation which include the second order terms to a certain extent, he also showed that when a soliton of the amplitude H collides with a vertical wall, the maximum wave height may be given by the following equation;

$$\eta\,\mathrm{max} = 2H + \frac{1}{2}H^2 + O(H^3)$$

The authors' result of calculation shown by Fig. 9 were not in good agreement with the result of analytical solution.
This may be attributed to improper selection of the initial value and use of different equations. In general Boussinesqs equation is slightly different from the KdV equation although both of them may have a soliton solution.
The result of numerical calculation showed that the reflected have is produced by collision. It is considered that production of the reflected wave may account for reduction of the amplitude of a given soliton due to collision. For more detailed analysis it may be necessary to use the finer mesh division and the smaller time interval.
T. Takutani (1971) and R.S. Johnson (1972) derived the KdV equation for the dispersive water wave propagating along the uneven bottom. In this case, the boundary condition at the bottom is different from eq. (6) and is given by the following equation;

$$\frac{\partial \phi}{\partial y} = \frac{\partial B}{\partial x}\frac{\partial \phi}{\partial x} \qquad \text{at } y = B(x) \qquad (48)$$

where B(x) is the bottom surface.
Introducing an infinitesimal parameter ε which represents the dispersion and nonlinearity, the following transformation of coordinate system is applied to the problem considered;

$$\xi = \varepsilon^{1/2}\left(\int \frac{dx}{\sqrt{1-B(X)}} - t\right), \quad X = \varepsilon^{3/2} x \qquad (49)$$

and assuming development of dependent variable as given by eqs. (16) and (17), and using the following relation

$$d(X) = 1 - B(X) \qquad (50)$$

the following KdV equation for $\eta^{(1)}$, the first perturbation term of η, is obtained;

$$\frac{\partial \eta^{(1)}}{\partial X} + \frac{1}{4}\frac{\eta^{(1)}}{d}\frac{\partial d}{\partial X} + \frac{3}{2}\frac{\eta^{(1)}}{d^{3/2}}\frac{\partial \eta^{(1)}}{\partial \xi} + \frac{d}{6}\frac{\partial^3 \eta^{(1)}}{\partial \xi^3} = 0 \qquad (51)$$

applying the following transformation

$$\eta^{(1)} = d^{-1/4}\, h(X, \xi) \qquad (52)$$

to eq. (51), it is finally transformed in the following form;

$$\frac{\partial h}{\partial X} + \frac{3}{2} d^{-1/4} \; h \frac{\partial h}{\partial \xi} + \frac{1}{6} d^{1/2} \frac{\partial^3 h}{\partial \xi^3} = 0 \qquad (53)$$

It should be mentioned that each coefficient of eq. (53) is a function of X. Using eq. (53), R.S. Johnon (1972) showed by the theory and numerical experiments that when a soliton in the region of $d = 1$ travels up along the slope and enters into the region of $d = d_0$ (<1) n solitons may be produced in the region of d_0 where n is a integer and it is given by the following equation;

$$d_0 = [n(n+1)]^{-4/9}$$

Conclusion

From the discussion mentioned so far it can be concluded that as far as one dimensional problem is concerned it may be possible to predict realistic behavior in propagation of a tsunami. More precisely, for the early stage of a tsunami initiation the linear theory may be applicable, while as the wave propagates the nonlinearity and dispersion becomes balanced and during this stage the KdV equation may be used for prediction of the wave propagation.

In case where there exists continental shelves, eq. (53) may be adopted. It may be considered that depending upon the situation, these theories may be combined and matched appropriately in order to predict actual behavior of a tsunami with sufficient accuracy. However, the actual tsunami propagates on the surface of the earth in two dimensional manner and then phenomena of refraction as well reflection may exist, and consequently the propagation may be influenced by these phenomena in very complicated manner. Therefore prediction of the two dimensional wave propagation must be extremely difficult to compare with the one dimensional problem. For analysis of such a problem the finite element method may be particularly important and powerful.

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