APPLICATION OF FINITE ELEMENT METHOD TO NON-SYMMETRICAL PROBLEMS OF SOLIDS OF REVOLUTION

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1) Introduction

The numerical analysis of three-dimensional elastic bodies with arbitrary shape, load, and boundary condition has become possible by means of the finite element method.[(1), (2)] But, from the view point of the practical analysis, it must be noted that even with big computers the capacity of the computation is restricted to a rough mesh of element division. The simplest finite elements for three-dimensional solids in the form of tetrahedron have their stiffness matrices of a large dimension of (12 x 12). Further, by dividing the three-dimensional solids into appropriately fine mesh of tetrahedrons, the number of the nodes can easily reach to many thousands. Necessity of tremendous amount of computer storage and the capacity of solving large simultaneous equations limit the scale of the problems which can be treated within the range of realistic computing time.

However, if we confine the problem to the solids of revolution with their geometry symmetrical with respect to an axis, the treatment becomes much easier. For example, in the axisymmetrical problems such as a thick-walled cylinder subjected to inner pressure the necessary unknown displacements are equal to those in the two-dimensional problems and the equilibrium equations formulated for one section passing through the axis of revolution are enough for the solution.

Similarly, for the non-symmetrical loading cases of the bodies of revolution, if the external loads vary in a sinusoidal manner in the circumferencial direction, the equations derived for an arbitrary axial section would represent the equilibrium for the whole body, because the resulted stress distribution should have the same circumferencial variation. This approach by circular harmonies is suggested by the authors of the literature (3). It is anticipated that the Fourier expansion of the external loads will facilitate the solution of the non-symmetrical problems of bodies of revolution.

In the present paper, the stiffness matrix of a ring element with triangular cross-section is presented for the case in which the circumferencial variation of the nodal forces and displacements are given by circular harmonies. The derivation of the stiffness matrix will be discussed in the literature (4).

As the actual application, the result of analysis of a prestressed concrete pressure vessel subjected to seismic horizontal body force is presented.

2) Loads, displacements, stresses and strains

We define the loads, displacements and stresses of a three-dimensional elastic element in the cylindrical co-ordinate as shown in Fig. 1 and 2.

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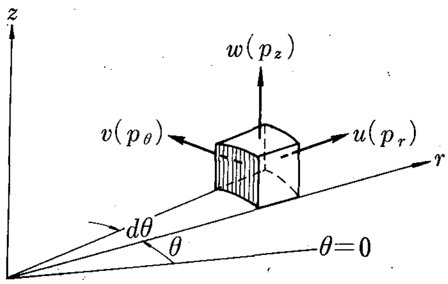

Fig. 1 Co-ordinate, displacements and (loads)

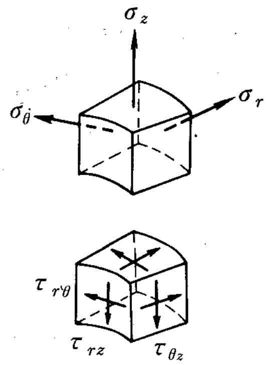

Fig. 2 Stresses

For the loads and displacements, we assume that they vary according to trigonometric functions. They can be written as

$$\left.\begin{aligned} \boldsymbol{p} &= \begin{bmatrix} p_r \\ p_\theta \\ p_z \end{bmatrix} = \begin{bmatrix} p_{rn}\cos n\theta \\ p_{\theta n}\sin n\theta \\ p_{zn}\cos n\theta \end{bmatrix} = \boldsymbol{T}_3\, \boldsymbol{p}_n \\ \boldsymbol{d} &= \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u_n \operatorname{con} n\theta \\ v_n \sin n\theta \\ w_n \cos n\theta \end{bmatrix} = \boldsymbol{T}_3\, \boldsymbol{d}_n \end{aligned}\right\} \quad \cdots\cdots(1)$$

where, n is positive integer and

$$\boldsymbol{T}_3 = diag. \{\cos n\theta \quad \sin n\theta \quad \cos n\theta\} = \begin{bmatrix} \cos n\theta & 0 & 0 \\ 0 & \sin n\theta & 0 \\ 0 & 0 & \cos n\theta \end{bmatrix} \quad \cdots\cdots(2)$$

We take, as the basis of the analysis, a ring element with triangular cross-section, the vertices of which are notated as i, j and k. The following expression gives the notation of the nodal displacements and the nordal forces of the element, as well as the assumption of their circumferential variation:

$$\begin{aligned} \boldsymbol{d}_e &= \{u_i\ u_j\ u_k\ ;\ v_i\ v_j\ v_k\ ;\ w_i\ w_j\ w_k\} \\ &= \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_n \cos n\theta \\ \boldsymbol{v}_n \sin n\theta \\ \boldsymbol{w}_n \cos n\theta \end{bmatrix} = \boldsymbol{T}_9\, \boldsymbol{d}_{en} \end{aligned} \quad \cdots\cdots(3)$$

$$\begin{aligned} \boldsymbol{f}_e &= \{f_{ri}\ f_{rj}\ f_{rk}\ ;\ f_{\theta i}\ f_{\theta j}\ f_{\theta k}\ ;\ f_{zi}\ f_{zj}\ f_{zk}\} \\ &= \begin{bmatrix} \boldsymbol{f}_r \\ \boldsymbol{f}_\theta \\ \boldsymbol{f}_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_{rn} \cos n\theta \\ \boldsymbol{f}_{\theta n} \sin n\theta \\ \boldsymbol{f}_{zn} \cos n\theta \end{bmatrix} = \boldsymbol{T}_9 \boldsymbol{f}_{en} \end{aligned} \quad \cdots\cdots(4)$$

where, $T_9 = diag.\{\cos n\theta \; \cos n\theta \; \cos n\theta \; \sin n\theta \; \sin n\theta \; \sin n\theta \; \cos n\theta \; \cos n\theta \; \cos n\theta\}$ ················(5)

Here the nodal forces must be taken as line forces on the respective ridge lines, their unit being such as t/m. ----- Fig. 3 and 4.

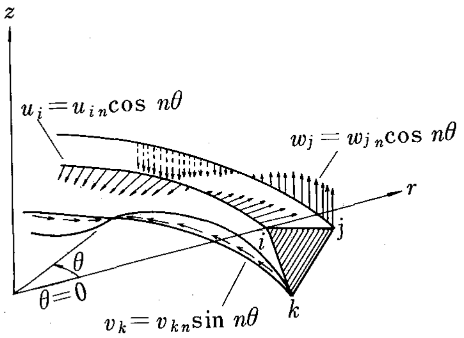

Fig.3 Nodal displacements

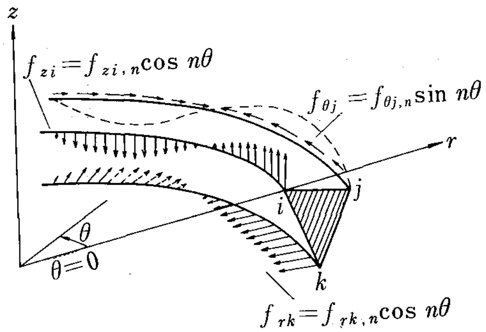

Fig. 4 Nodal forces

3) Expression for strains and stresses of the element

Assuming linear distribution of stresses within the triangular cross-section and using the notation

$$A = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_k & z_k \end{bmatrix}^{-1} \qquad \cdots\cdots(6)$$

$$a = [\alpha + \beta r + \gamma z] \qquad \cdots\cdots(7)$$

where (r_i, z_i), (r_j, z_j) and (r_k, z_k) are the co-ordinates of the each vertex, the strains within the ring element caused by the unit nodal displacements are expressed by a (9 x 6) matrix ε :

$$\bar{E} = [\bar{\varepsilon}_{ui} \; \bar{\varepsilon}_{uj} \; \bar{\varepsilon}_{uk} \; ; \; \bar{\varepsilon}_{vi} \; \bar{\varepsilon}_{vj} \; \bar{\varepsilon}_{vk} \; ; \; \bar{\varepsilon}_{wi} \; \bar{\varepsilon}_{wj} \; \bar{\varepsilon}_{wk}]$$

$$= \begin{array}{c} \\ \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{rz} \\ \gamma_{r\theta} \\ \gamma_{\theta z} \end{array} \begin{bmatrix} u & v & w \\ \beta\cos n\theta & 0 & 0 \\ \frac{1}{r}a\cos n\theta & \frac{n}{r}a\cos n\theta & 0 \\ 0 & 0 & \gamma\cos n\theta \\ \gamma\cos n\theta & 0 & \beta\cos n\theta \\ -\frac{n}{r}a\sin n\theta & (\beta-\frac{a}{r})\sin n\theta & 0 \\ 0 & \gamma\sin n\theta & -\frac{n}{r}a\sin n\theta \end{bmatrix} \qquad \cdots\cdots(8)$$

Here, for example, the column vector ε_{ui} means the strains $\{\varepsilon_r \; \varepsilon_\theta \; \varepsilon_z \; \gamma_{rz} \; \gamma_{r\theta} \; \gamma_{\theta z}\}$ caused by the unit displacement $u_i = \cos n\theta$. According to the three-dimensional theory of elasticity, the stresses of the element are

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \\ \tau_{r\theta} \\ \tau_{\theta z} \end{bmatrix} = \mathbb{D}\boldsymbol{\varepsilon} = \mathbb{D}\,\bar{\mathbb{E}}\;den \qquad \cdots\cdots (9)$$

where, for the general cases,

$$\mathbb{D} = \begin{bmatrix} E_{12} & E_{12} & E_{13} & 0 & 0 & 0 \\ & E_{22} & E_{23} & 0 & 0 & 0 \\ & & E_{33} & 0 & 0 & 0 \\ & & & G_{44} & 0 & 0 \\ & \text{symmetric} & & & G_{55} & 0 \\ & & & & & G_{66} \end{bmatrix} \qquad \cdots\cdots (10)$$

and for the isotropic bodies,

$$\mathbb{D} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ & & 2\mu+\lambda & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & \text{symmetric} & & & \mu & 0 \\ & & & & & \mu \end{bmatrix} \qquad \cdots\cdots (11)$$

with, $\mu = \dfrac{E}{2(1+\nu)}$, $\lambda = \dfrac{2\nu}{1-2\nu}\mu$; $\qquad \cdots\cdots (12)$

: Young's modulus,
: Poisson's ratio.

4) Stiffness matrix of the element

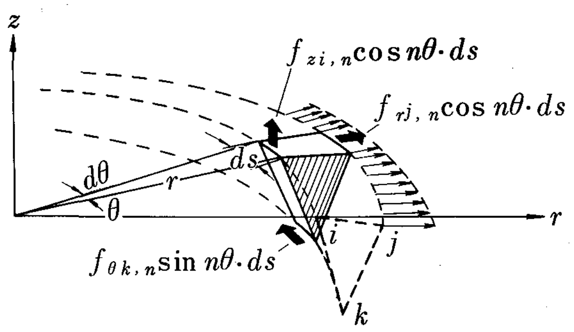

Fig. 5 Nodal forces at an arbitrary location in a ring element

The application of the principle of virtual work to a portion sectorial in plan between the angles θ and $\theta + d\theta$ shown in Fig. 5 leads to the equilibrium equation between stresses and nodal forces.

From this equilibrium equation, we finally reach to the force-displacement relation of the form

$$r\, \boldsymbol{f}_{en} = \boldsymbol{K}_{en}\, \boldsymbol{d}_{en} \qquad \cdots\cdots (13)$$

$\boldsymbol{K}_{en}$ is the stiffness matrix which defines the relation between the maximum values of the nodal forces and the nodal displacements:

$$\boldsymbol{K}_{en}\,(9\times 9) = \begin{bmatrix} \boldsymbol{K}_{11} & \boldsymbol{K}_{12} & \boldsymbol{K}_{13} \\ & \boldsymbol{K}_{22} & \boldsymbol{K}_{23} \\ \text{symmetric} & & \\ & & \boldsymbol{K}_{33} \end{bmatrix} \qquad \cdots\cdots (14)$$

its elements $\boldsymbol{K}_{ij}$ being obtained as follows:

$$\left.\begin{aligned}
\boldsymbol{K}_{11} &= I_1\{(E_{11}+2E_{12}+E_{22}+n^2G_{55})\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\beta}+G_{44}\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\gamma}\}+(E_{12}+E_{22}+n^2G_{55}) \\
&\quad\times(I_2\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\alpha}+I_3\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\gamma})+(E_{12}+E_{22}+n^2G_{55})(I_2\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\beta}+I_3\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\beta})+(E_{22}+n^2G_{55}) \\
&\quad\times\{I_4\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}+I_5(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\gamma}+\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\alpha})+I_6\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\gamma}\}, \\
\boldsymbol{K}_{12} &= n[I_1(E_{12}+E_{22})\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\beta}+(E_{12}+E_{22}+G_{55})(I_2\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\alpha}+I_3\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\gamma})+E_{22}(I_2\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\beta} \\
&\quad+I_3\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\beta})+(E_{22}+G_{55})\{I_4\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}+I_5(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\gamma}+\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\alpha})+I_6\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\gamma}\}], \\
\boldsymbol{K}_{13} &= I_1\{(E_{13}+E_{23})\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\gamma}+G_{44}\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\beta}\}+E_{23}(I_2\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\gamma}+I_3\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\gamma}), \\
\boldsymbol{K}_{22} &= I_1(n^2E_{22}\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\beta}+G_{66}\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\gamma})+n^2E_{22}\{I_2(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\beta}+\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\alpha})+I_3(\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\gamma}+\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\beta})\} \\
&\quad+(n^2E_{22}+G_{55})\{I_4\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}+I_5(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\gamma}+\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\alpha})+I_6\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\gamma}\} \\
\boldsymbol{K}_{23} &= n\{I_1(E_{23}\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\gamma}-G_{66}\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\beta})-G_{66}(I_2\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\alpha}+I_3\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\gamma})+E_{23}(I_2\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\gamma}+I_3\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\gamma})\} \\
\boldsymbol{K}_{33} &= I_1\{E_{33}\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\gamma}+(G_{44}+n^2G_{66})\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\beta}\}+n^2G_{66}\{I_2(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\beta}+\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\alpha})+I_3(\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\gamma}+\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\beta})\} \\
&\quad+n^2G_{66}\{I_4\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}+I_5(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\gamma}+\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\alpha})+I_6\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\gamma}\}
\end{aligned}\right\} \quad (15)$$

where $I_1, I_2, \ldots$ and I_6 are the integration terms

$$\left.\begin{aligned}
I_1 &= \iint_{\Delta} r\,dr\,dz = \bar{r}A, \\
I_2 &= \iint_{\Delta} dr\,dz = A, \\
I_3 &= \iint_{\Delta} z\,dr\,dz = \bar{z}A, \\
I_4 &= \iint_{\Delta} \frac{1}{r}\,dr\,dz \doteqdot \frac{1}{\bar{r}}A, \\
I_5 &= \iint_{\Delta} \frac{z}{r}\,dr\,dz \doteqdot \frac{\bar{z}}{\bar{r}}A, \\
I_6 &= \iint_{\Delta} \frac{z^2}{r}\,dr\,dz \doteqdot \frac{1}{3\bar{r}}\left\{\left(\frac{z_i+z_j}{2}\right)^2+\left(\frac{z_j+z_k}{2}\right)^2+\left(\frac{z_k+z_i}{2}\right)^2\right\}A,
\end{aligned}\right\} \quad \cdots\cdots (16)$$

and

$$\left.\begin{aligned}
\bar{r} &= \frac{1}{3}(r_i+r_j+r_k), \\
\bar{z} &= \frac{1}{3}(z_i+z_j+z_k), \\
A &= \frac{1}{2}\begin{vmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_k & z_k \end{vmatrix}
\end{aligned}\right\} \quad \cdots\cdots (17)$$

The approximate expressions for I_4, I_5 and I_6 are valid under the assumption that the dimension of the cross-section of the element is small compared to its radius of revolution

5) Example-stress distribution of a prestressed concrete pressure vessel subjected to seismic lateral force

As an application for the case n=1, a prestressed concrete pressure vessel of cylindrical type under seismic horizontal body force was analysed. The computation was carried out by means of the overrelaxated iteration.

Fig.6 shows the dimension of the vessel, the idealization by triangular mesh and the assumed acceleration. The distribution of displacements, stresses at the section $\theta=0$ or $\theta=\frac{\pi}{2}$ are given in the following figures.

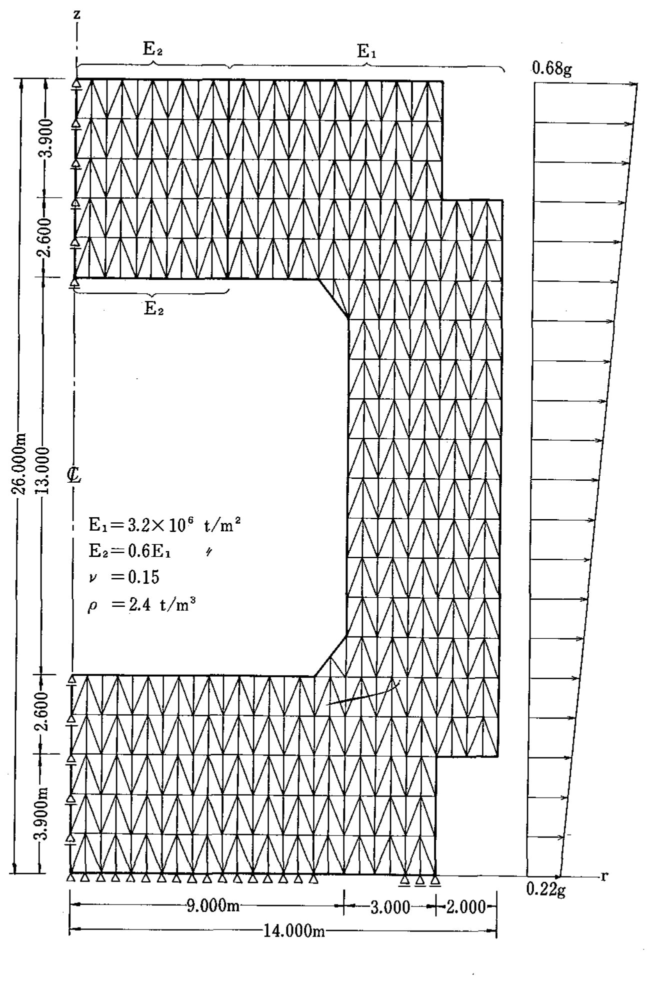

Fig.6 Dimension, finite element idealization and acceleration

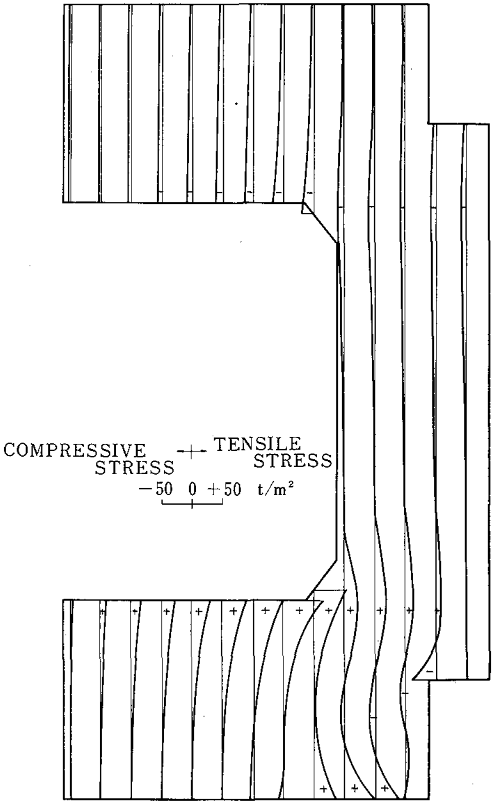

Fig. 7 Stress (σ_r) in θ = 0 section

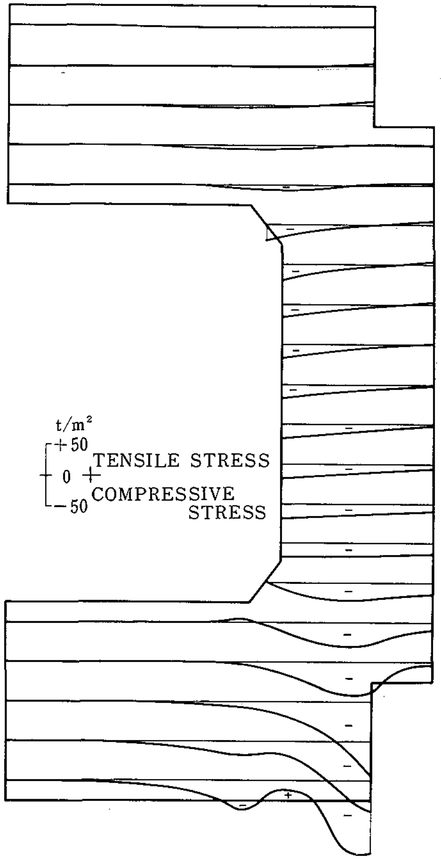

Fig. 8 Stress (σ_z) in θ = 0 section

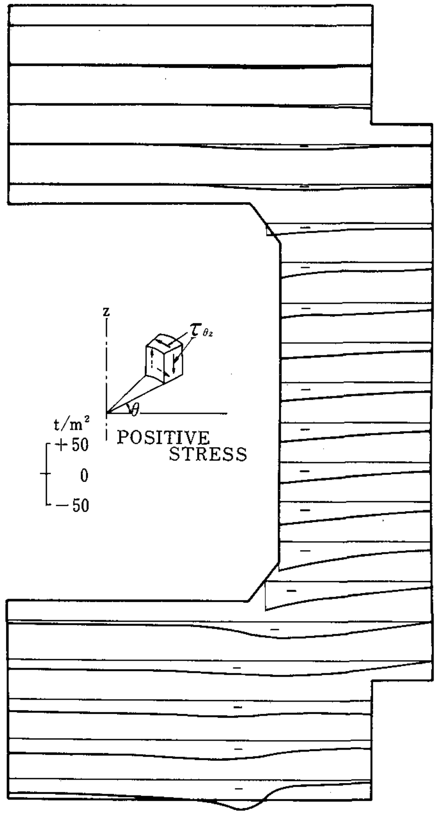

Fig. 9 Stress ($\tau_{\theta z}$) in $\theta = \frac{\pi}{2}$ section

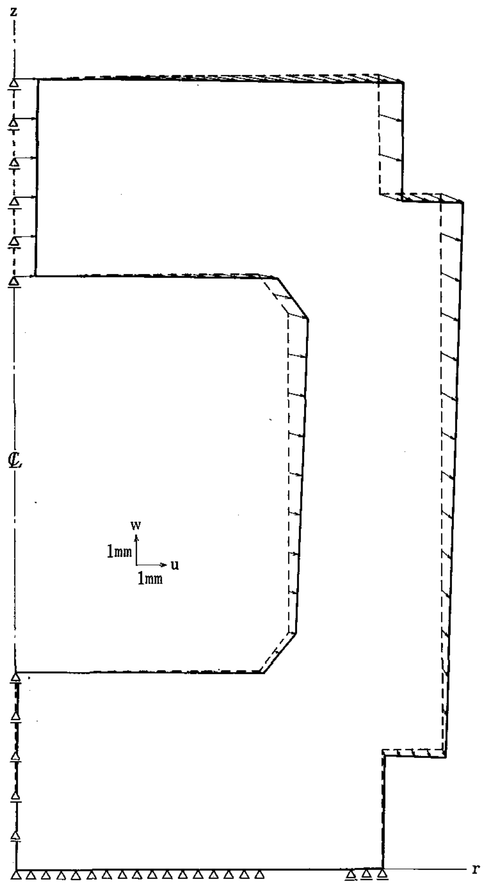

Fig. 10 Displacement in $\theta = 0$ section

Literatures

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